Math 7800 Worksheet #6

Due April 17th in gradescope

1.

Suppose that $\pi : \widetilde{X} \longrightarrow X, \pi' : \widetilde{X}' \longrightarrow X$ are two log resolutions of a Noetherian equidimensional scheme X of finite type and separated over a field of characteristic zero, with reduced exceptional divisors F, F' respectively. Show that $\pi_* \omega_{\widetilde{X}}(F) = \pi'_* \omega_{\widetilde{X}'}(F')$.

Hint: Show that we may assume that we may factor our resolutions as follows: $\pi' : \widetilde{X}' \xrightarrow{\rho} \widetilde{X} \xrightarrow{\pi} X$. Notice that we do not necessarily have $F' = \rho^{-1}(F)_{\mathrm{red}}$ but we do have $F' \supseteq \rho^{-1}(F)_{\mathrm{red}}$. Now use the projection formula and the fact that $\mathcal{J}(\widetilde{X}, (1-\epsilon)F) \cdot \omega_{\widetilde{X}} = \omega_{\widetilde{X}}$ for $1 \gg \epsilon > 0$ (which can be deduced either directly, or via reduction to characteristic p).

2.

Suppose that $X \subseteq Y$ is a closed embedding of varieties over a field k of characteristic zero and that Y is smooth. Further suppose X is non-singular except at a single closed point $x \in X$, that $\pi : Y' \longrightarrow Y$ is an embedded log resolution of singularities of X which is an isomorphism except over x and $E = \pi^{-1}(x)_{\mathrm{red}}$ is a SNC divisor. Further set $\widetilde{X}$ be the strict transform of X. Prove that there exists an exact triangle in the derived category:

$$\underline{\Omega}^0_X \longrightarrow \mathbf{R}\pi_* \mathcal{O}_{\widetilde{X}} \oplus k(x) \longrightarrow \mathbf{R}\pi_* \mathcal{O}_{E \cap \widetilde{X}} \xrightarrow{+1} .$$

Here you should define $\underline{\Omega}^0_X := \mathbf{R}\pi_* \mathcal{O}_{\overline{X}}$ where $\overline{X} = (\pi^{-1}(X))_{\mathrm{red}}$. In fact, there are a number of generalizations of this when x is not a point.

Hint: Recall that $\pi^{-1}(X)_{\mathrm{red}} = \overline{X} = \widetilde{X} \cup E$ is such that $\mathbf{R}\pi_* \mathcal{O}_{\overline{X}} = \underline{\Omega}^0_X$. Construct a related short exact sequence on Y' and then push it down. Finally, this is easier if you assume that k is algebraically closed, so if you are stuck on the general case, assume that.

3.

Suppose that Z is a smooth projective variety in characteristic zero. Then for any ample line bundle $\mathscr{L}$ on Z and any sufficiently large integer $m \gg 0$, we have that the section ring with respect to $\mathscr{L}^m$,

$$S := \bigoplus_{n \in \mathbb{Z}} H^0(Z, \mathscr{L}^{nm}),$$

has Du Bois singularities. In fact, this holds under the weaker hypothesis that Z has Du Bois singularities even if Z is not smooth.

Hint: Use the previous exercise. Note that if $x = \operatorname{Spec} S$ and $\pi : \widetilde{X} \longrightarrow X$ is the blowup of the cone point with exceptional divisor $F \cong Z$, then it suffices to show that $\mathbf{R}^i \pi_* \mathscr{O}_{\widetilde{X}} \longrightarrow \mathbf{R}^i \mathscr{O}_F$ is an isomorphism for $i > 0$. Prove this isomorphism degree by degree (the degree zero piece is the only one which requires real computation).

4.

Suppose R is an integral domain and $Q \subseteq R$ is a prime ideal. Show that for any prime $Q^+ \subseteq R^+$ lying over Q (which necessarily exist since $R \subseteq R^+$ is integral) we have that

$$R^+/Q^+ \cong (R/Q)^+.$$

5.

Let R be a domain containing a field of characteristic zero. Prove that $I^+ = IS \cap R$ where S is the normalization of R. (Recall $I^+ = IR^+ \cap R$).

Hint: Use the field trace to show that if S is a normal domain and T is any module finite extension of S, then the inclusion $S \hookrightarrow T$ splits as an S-module map.

6.

Let R be an F-finite Noetherian local domain of prime characteristic p with system of parameters $x_1, \dots, x_d$. Show that an element $\eta = [z + (x_1, \dots, x_d)] \in H^d_{\mathfrak{m}}(R)$ is in the plus closure of zero if and only if $z \in (x_1, \dots, x_d)^+$.

I asked this question to both Claude Opus 4.6 and ChatGPT Pro on 4/6/2026. Compare the two responses and decide if they are both correct / which is better.

Here is Claude's response.

We write $\underline{x} = x_1, \dots, x_d$ and $I = (x_1, \dots, x_d)$. Throughout, we use the Čech cohomology description

$$H^d_{\mathfrak{m}}(R) \;\cong\; \frac{R_{x_1 \cdots x_d}}{\displaystyle\sum_{i=1}^{d} R_{x_1 \cdots \hat{x}_i \cdots x_d}},$$

under which η corresponds to the class $\left[\dfrac{z}{x_1 \cdots x_d}\right]$.

Proof. **(⇐):** Suppose $z \in I^+$. Then there exists a module-finite extension domain $R \hookrightarrow S$ with $z \in IS$, say $z = \sum_{i=1}^d s_i x_i$ for some $s_i \in S$. Since S is module-finite over R, the ideal IS is $\mathfrak{m}S$-primary, and the Čech complex on $\underline{x}$ computes $H^d_{\mathfrak{m}S}(S)$. The image of η in $H^d_{\mathfrak{m}S}(S)$ is represented by

$$\frac{z}{x_1 \cdots x_d} \;=\; \sum_{i=1}^{d} \frac{s_i\, x_i}{x_1 \cdots x_d} \;=\; \sum_{i=1}^{d} \frac{s_i}{x_1 \cdots \hat{x}_i \cdots x_d} \;\in\; \sum_{i=1}^{d} S_{x_1 \cdots \hat{x}_i \cdots x_d}.$$

Hence the class is zero in $H^d_{\mathfrak{m}S}(S)$, so $\eta \in 0^+_{H^d_{\mathfrak{m}}(R)}$.

(⇒): Suppose $\eta \in 0^+_{H^d_{\mathfrak{m}}(R)}$. Then there exists a module-finite extension domain $R \hookrightarrow S$ such that $\left[\dfrac{z}{x_1 \cdots x_d}\right] = 0$ in $H^d_{\mathfrak{m}S}(S)$. Vanishing in Čech cohomology gives

$$\frac{z}{x_1 \cdots x_d} \;=\; \sum_{i=1}^{d} \frac{a_i}{(x_1 \cdots \hat{x}_i \cdots x_d)^{n_i}}$$

for some $a_i \in S$ and $n_i \geq 1$, as elements of $S_{x_1 \cdots x_d}$. Clearing denominators by multiplying through by a sufficiently high power $(x_1 \cdots x_d)^N$ (with N large enough), we obtain a relation in S of the form

$$z \cdot (x_1 \cdots x_d)^{N-1} \;\in\; (x_1^N, \dots, x_d^N)\, S.$$

Now let R^+ denote the absolute integral closure of R (the integral closure of R in an algebraic closure of its fraction field). Since R is an excellent local domain of prime characteristic, R^+ is a big Cohen–Macaulay algebra; that is, every system of parameters for R is a regular sequence on R^+. In particular, $x_1^N, \dots, x_d^N$ is a system of parameters for R, so it forms a regular sequence on R^+.

Since S is a module-finite extension domain of R, we have $S \subseteq R^+$, and thus the relation above holds in R^+:

$$z \cdot (x_1 \cdots x_d)^{N-1} \;\in\; (x_1^N, \dots, x_d^N)\, R^+.$$

Because $x_1^N, \dots, x_d^N$ is a regular sequence on R^+, the ideal $(x_1^N, \dots, x_d^N)R^+$ satisfies the standard monomial property: an element r satisfies $r \cdot x_1^{a_1} \cdots x_d^{a_d} \in (x_1^N, \dots, x_d^N)R^+$ with each $a_i < N$ only if $r \in (x_1^N, \dots, x_d^N)R^+$ itself. More precisely, since $x_1^N, \dots, x_d^N$ is a regular sequence on R^+, the associated graded ring (or equivalently, the Koszul homology) detects membership: by iterating the regular sequence property one variable at a time, we deduce that

$$z \in (x_1, \dots, x_d)\, R^+.$$

Lemma. *If $x_1^N, \dots, x_d^N$ is a regular sequence on a ring A and $z \cdot (x_1 \cdots x_d)^{N-1} \in (x_1^N, \dots, x_d^N)A$, then $z \in (x_1, \dots, x_d)A$.*

Proof of Lemma. We induct on d. When $d = 1$, the hypothesis gives $z\, x_1^{N-1} \in (x_1^N)A$, i.e. $x_1^N \mid z\, x_1^{N-1}$ in A. Since x_1 is a non-zero-divisor, $x_1 \mid z$, so $z \in (x_1)A$.

For $d > 1$, write $P = (x_1 \cdots x_{d-1})^{N-1}$. The hypothesis is

$$z \cdot P \cdot x_d^{N-1} \in (x_1^N, \dots, x_{d-1}^N, x_d^N)\, A.$$

Working modulo $(x_1^N, \dots, x_{d-1}^N)A$, we get

$$z \cdot P \cdot x_d^{N-1} \in (x_d^N)\, \overline{A},$$

where $\overline{A} = A/(x_1^N, \dots, x_{d-1}^N)A$. Since $x_1^N, \dots, x_d^N$ is a regular sequence, x_d^N (hence also x_d) is a non-zero-divisor on $\overline{A}$. Thus $x_d \mid z \cdot P$ in $\overline{A}$, which iterated $N - 1$ times gives $x_d^{N-1} \mid z \cdot P$ in $\overline{A}$. That is,

$$z \cdot P = x_d^{N-1}\, w + q, \qquad q \in (x_1^N, \dots, x_{d-1}^N)A$$

for some $w \in A$. Substituting back into the original relation shows $x_d^{N-1}(z\, P - x_d^{N-1}\, w) \in (x_1^N, \dots, x_{d-1}^N)A$, but this is exactly $x_d^{N-1}\, q' \in (x_1^N, \dots, x_{d-1}^N)A$ for suitable q'. Working modulo x_d, we see $z\, P \in (x_1^N, \dots, x_{d-1}^N, x_d)A$.

Since x_d is regular on $A/(x_1^N, \dots, x_{d-1}^N)$, we can rearrange: write $z\, P = \sum_{i=1}^{d-1} a_i x_i^N + b\, x_d$ for some $a_i, b \in A$. Now $x_1^N, \dots, x_{d-1}^N$ is a regular sequence on $A/x_d A$ (by permutability of regular sequences), and $z \cdot P \equiv 0 \pmod{(x_1^N, \dots, x_{d-1}^N)}$ in $A/x_d A$. By the inductive hypothesis applied to $A/x_d A$ (with the regular sequence $x_1^N, \dots, x_{d-1}^N$), we obtain $z \in (x_1, \dots, x_{d-1}, x_d)A$. ☐

Applying the lemma with $A = R^+$, we conclude $z \in (x_1, \dots, x_d)\, R^+ = I\, R^+$. Since R^+ is the directed union of all module-finite extension domains S of R, the relation $z \in I\, R^+$ must already hold in some module-finite extension domain S'. That is, $z \in I\, S'$ with $R \hookrightarrow S'$ a module-finite extension of domains, which gives $z \in I^+$. ☐

Here is ChatGPT's answer.

Let $I = (x_1, \dots, x_d)$ and $A = R^+$. Since I is $\mathfrak{m}$-primary,

$$H^d_{\mathfrak{m}}(R) \cong H^d_I(R).$$

By definition,

$$0^+_{H^d_{\mathfrak{m}}(R)} = \ker\Big(H^d_I(R) \longrightarrow H^d_I(A)\Big).$$

Also,

$$H^d_I(A) \cong \varinjlim_n A/(x_1^n, \dots, x_d^n)A,$$

where the transition maps are multiplication by $x_1 \cdots x_d$. Hence the image of

$$\eta = [z + I] \in H^d_I(R)$$

in $H^d_I(A)$ is zero if and only if for some $n \geq 0$,

$$(x_1 \cdots x_d)^n z \in (x_1^{n+1}, \dots, x_d^{n+1})A. \tag{$\ast$}$$

If $z \in I^+ = IA \cap R$, write $z = \sum_i x_i a_i$ with $a_i \in A$. Then

$$(x_1 \cdots x_d)^n z = \sum_i x_i^{n+1} \Big(\prod_{j \neq i} x_j^n \Big) a_i \in (x_1^{n+1}, \dots, x_d^{n+1})A,$$

so $(\ast)$ holds and therefore $\eta \in 0^+_{H^d_{\mathfrak{m}}(R)}$.

Conversely, assume $(\ast)$. Since $A = R^+$ is Cohen–Macaulay, every system of parameters on R is an A-regular sequence; in particular every permutation of $x_1, \dots, x_d$ is regular on A. Set

$$u_t = (x_1 \cdots x_t)^n z \qquad (0 \leq t \leq d),$$

so $\nu_d \in (x_1^{n+1}, \dots, x_d^{n+1})A$ by $(\ast)$. We claim, descending on t, that

$$\nu_t \in (x_1^{n+1}, \dots, x_t^{n+1}, x_{t+1}, \dots, x_d)A. \tag{$\dagger_t$}$$

This is true for $t = d$. Suppose $(\dagger_t)$ holds with $t \geq 1$. Mod out by

$$(x_1^{n+1}, \dots, x_{t-1}^{n+1}, x_{t+1}, \dots, x_d)A.$$

In that quotient, x_t is a nonzerodivisor, and $(\dagger_t)$ becomes

$$x_t^n \nu_{t-1} \in x_t^{n+1} A.$$

Cancelling x_t^n gives

$$\nu_{t-1} \in (x_1^{n+1}, \dots, x_{t-1}^{n+1}, x_t, \dots, x_d)A,$$

which is $(\dagger_{t-1})$. Iterating down to $t = 0$ yields

$$z = \nu_0 \in (x_1, \dots, x_d)A.$$

Since $z \in R$, this says

$$z \in (x_1, \dots, x_d)A \cap R = (x_1, \dots, x_d)^+.$$

Therefore

$$\eta \in 0^+_{H^d_{\mathfrak{m}}(R)} \iff z \in (x_1, \dots, x_d)^+.$$

7.

Suppose $(R, \mathfrak{m})$ is a Noetherian local domain and $0 \neq f \in \mathfrak{m}$. Show that if $R/(f)$ is BCM-rational, then R is also BCM-rational.

Hint: R is certainly Cohen-Macaulay. Next observe that for any balanced big Cohen-Macaulay R-algebra B, we have that $R/(f)$ is balanced big Cohen-Macaulay over $R/(f)$. Use a strategy similar to the rational case.

8.

A domain R is called a *derived splinter* if for every proper surjective map $\pi : X \longrightarrow \operatorname{Spec} R$ we have that $\mathcal{O}_{\operatorname{Spec} R} \longrightarrow \mathbf{R}\pi_* \mathcal{O}_X$ splits (in the derived category). Prove that a strongly F-regular ring is a derived splinter.