

Math 7800 Worksheet #5

Due April 3rd in gradescope

1.

Let (S, \mathfrak{m}) be an F -finite regular local ring, and suppose that $R = S/(f_1, \dots, f_r)$ is a complete intersection.¹ Prove that R is Frobenius split if and only if

$$(f_1 \cdots f_r)^{p-1} \notin \mathfrak{m}^{[p]}.$$

This makes checking Fedder's criterion for complete intersections almost as easy as checking Fedder's criterion for hypersurfaces.

I asked Utah's ChatGPT Pro this question on 3/22/2026. This is its response after thinking for about 15 minutes (I asked it to keep it concise and convert its answer to LaTeX). Determine if it correct. If it is not correct, identify the errors and write a correct proof.

Chat GPT's Solution:

Proof. Let $I = (f_1, \dots, f_r)$, so $R = S/I$. Since S is F -finite, so is R , and for F -finite rings, F -purity is equivalent to Frobenius splitting. Thus Fedder's criterion gives

$$R \text{ is Frobenius split} \iff (I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}.$$

So it remains to compute $(I^{[p]} : I)$.

We claim that if x_1, \dots, x_t is a regular sequence and $J_t = (x_1^p, \dots, x_t^p)$, then

$$(J_t : (x_1, \dots, x_t)) = J_t + ((x_1 \cdots x_t)^{p-1}).$$

We prove this by induction on t . For $t = 1$,

$$(x_1^p : x_1) = (x_1^{p-1}).$$

Now assume $t > 1$, and write $J' = (x_1^p, \dots, x_{t-1}^p)$, so $J_t = J' + (x_t^p)$. Let

$$u \in (J_t : (x_1, \dots, x_t)).$$

Since $ux_t \in J_t$ and x_t is a nonzerodivisor modulo J' , we get

$$u \in J' + (x_t^{p-1}),$$

so $u \equiv vx_t^{p-1} \pmod{J'}$ for some $v \in S$. For each $i < t$, the condition $ux_i \in J_t$ implies

$$vx_i \in J' + (x_t).$$

¹Complete intersection means that $\dim R = \dim S - r$, or equivalently, that f_1, \dots, f_r form a regular sequence.

Reducing modulo x_t , this says

$$\bar{v} \in \left((\bar{x}_1^p, \dots, \bar{x}_{t-1}^p) : (\bar{x}_1, \dots, \bar{x}_{t-1}) \right).$$

Since $\bar{x}_1, \dots, \bar{x}_{t-1}$ is again a regular sequence in $S/(x_t)$, the induction hypothesis yields

$$\bar{v} \in (\bar{x}_1^p, \dots, \bar{x}_{t-1}^p, (\bar{x}_1 \cdots \bar{x}_{t-1})^{p-1}).$$

Lifting back to S ,

$$v \in J' + (x_t) + \left((x_1 \cdots x_{t-1})^{p-1} \right),$$

hence

$$u \in J_t + \left((x_1 \cdots x_t)^{p-1} \right).$$

The reverse inclusion is immediate, since

$$(x_1 \cdots x_t)^{p-1} x_i \in J_t \quad \text{for all } i.$$

This proves the claim.

Applying the claim to $x_i = f_i$, we obtain

$$(I^{[p]} : I) = I^{[p]} + \left((f_1 \cdots f_r)^{p-1} \right).$$

Because $I \subseteq \mathfrak{m}$, we have $I^{[p]} \subseteq \mathfrak{m}^{[p]}$. Therefore

$$(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]} \iff (f_1 \cdots f_r)^{p-1} \notin \mathfrak{m}^{[p]}.$$

Combining this with Fedder's criterion finishes the proof:

$$R \text{ is Frobenius split} \iff (f_1 \cdots f_r)^{p-1} \notin \mathfrak{m}^{[p]}. \quad \square$$

What do you think?

2.

Suppose that R is a regular Noetherian ring of characteristic $p > 0$ and that \mathfrak{q} is a prime ideal. Prove that $\mathfrak{q}^{[p^e]}$ is \mathfrak{q} -primary.

Hint: Show that if $f \notin \mathfrak{q}$, then the “multiplication by f ” map $R/\mathfrak{q}^{[p^e]} \xrightarrow{f} R/\mathfrak{q}^{[p^e]}$ is injective.

3.

Suppose that S is an F -finite regular ring that is *not necessarily local* and $R = S/I$. Fix a prime ideal $\mathfrak{q} \in \text{Spec } R = \mathbb{V}(I) \subseteq \text{Spec } S$. Show that $\text{Spec } R$ is Frobenius split at \mathfrak{q} if and only if for some fixed (equivalently, all) $e \in \mathbb{N}$,

$$(I^{[p^e]} : I) \not\subseteq \mathfrak{q}^{[p^e]}.$$

Hint: Reduce to the case where \mathfrak{q} is the maximal ideal of a regular local ring using the previous exercise to show that $\mathfrak{q}^{[p^e]}R_{\mathfrak{q}} \cap R = \mathfrak{q}^{[p^e]}$.

4.

Suppose S is an F -finite regular ring and $F_*^e S$ has an S -basis $F_*^e b_1, \dots, F_*^e b_m$. Further suppose that $\Phi^e \in \text{Hom}_S(F_*^e S, S)$ is a generating map. Suppose $f \in S$ and write

$$F_*^e f = \sum_{i=1}^m a_i F_*^e b_i = F_*^e \sum_{i=1}^m a_i^{p^e} b_i.$$

Then

$$\Phi^e(F_*^e f S) = (a_1, \dots, a_m).$$

5.

- (a) Suppose S is an F -finite ring with $\Phi^e \in \text{Hom}_S(F_*^e S, S)$ a generating map. For any ideal $J \subseteq S$, prove that $\Phi^e(F_*^e J)$ is the same as the image of

$$(F_*^e J) \cdot \text{Hom}_S(F_*^e S, S) \xrightarrow{\text{eval}@F_*^e 1} S$$

- (b) Consider the ring $R = \mathbb{F}_3[x, y]/(x^2 + y^5)$. Let J_e be the image of “evaluation at $F_*^e 1$ ” map $\text{Hom}_R(F_*^e R, R) \rightarrow R$. Compute the ideals J_1 and J_2 , and show they are not equal, but do define the same locus.

6.

Let A be a domain finitely generated over \mathbb{Z} and contained in a field k of characteristic zero. Prove that if \mathcal{U} is a non-empty open set of $\text{m-Spec}R$, then among the fields $\{A/\mu \mid \mu \in \mathcal{U}\}$, there are fields of every positive characteristic p except for finitely many p .

Hint: For $b \in A \setminus \{0\}$, consider $bA \cap \mathbb{Z}$.

7.

Suppose Y is a scheme essentially of finite type over a field of characteristic zero and $\pi : Y \rightarrow X$ is a resolution of singularities. Let \mathcal{M} be a finite rank locally free sheaf on X . Suppose that X has rational singularities and use that to show that

$$H^i(X, \mathcal{M}) \cong H^i(Y, \pi^* \mathcal{M})$$

for every integer i .

Hint: Observe that $\mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{O}_X} \mathbf{R}\pi_* \mathcal{O}_Y$ and then use the derived projection formula. While you can use spectral sequences, composition of derived functors is probably easier.

8.

Suppose that (R, \mathfrak{m}) is a reduced equidimensional local ring essentially of finite type over a field of characteristic zero. Suppose that $f \in \mathfrak{m}$ is a non-zerodivisor such that $R/(f)$ has rational singularities. Prove that R has rational singularities. This was first shown by R. Elkik in 1978.

Hint: First show that R is Cohen-Macaulay. Let $\pi : Y \rightarrow X = \operatorname{Spec} R$ be a resolution of singularities that is also an embedded resolution of the divisor $D = \operatorname{Div}(f)$. Let \tilde{D} denote the strict transform of D . Push forward the short exact sequence $0 \rightarrow \omega_Y \rightarrow \omega_Y(\tilde{D}) \rightarrow \omega_{\tilde{D}} \rightarrow 0$, using Grauert-Riemenschneider vanishing, compare with $0 \rightarrow \omega_X \rightarrow \omega_X(D) \rightarrow \omega_D \rightarrow 0$ and use the snake lemma, or argue similarly to the F -injective case after Matlis duality.

9.

Suppose that X is a normal variety and $D \subseteq X$ is a reduced² divisor. Fix $\pi : Y \rightarrow X$ an embedded resolution of (X, D) with $\tilde{D} := \pi_*^{-1}D$ the strict transform of D . Show that $\pi_*\omega_Y(\tilde{D}) \subseteq \omega_X(D)$ is independent of the choice of embedded resolution Y .

Hint: See Chapter VI, 2.21 for independence of $\pi_*\omega_Y$.

I asked Utah's ChatGPT Pro this on 3/22/2026. This is what it said after thinking for about 23 minutes. Is it right? If not identify the errors and correct it.

Proof. Let $\pi_i : Y_i \rightarrow X$ ($i = 1, 2$) be two embedded resolutions of (X, D) , and let $\tilde{D}_i \subset Y_i$ be the strict transform of D . Choose a smooth variety Z dominating both Y_1 and Y_2 :

$$\begin{array}{ccc}
 & Z & \\
 f_1 \swarrow & & \searrow f_2 \\
 Y_1 & & Y_2 \\
 \pi_1 \searrow & & \swarrow \pi_2 \\
 & X &
 \end{array}$$

and let $\tilde{D}_Z \subset Z$ be the strict transform of D . It is enough to show that for any proper birational morphism $f : Z \rightarrow Y$ between embedded resolutions,

$$f_*\omega_Z(\tilde{D}_Z) \cong \omega_Y(\tilde{D}_Y).$$

Indeed, then

$$(\pi_i)_*\omega_{Y_i}(\tilde{D}_i) = (\pi_i \circ f_i)_*\omega_Z(\tilde{D}_Z) \quad (i = 1, 2),$$

so the two pushforwards agree.

Now fix such an $f : Z \rightarrow Y$. Since \tilde{D}_Z and \tilde{D}_Y are smooth divisors, we have exact sequences

$$0 \rightarrow \omega_Z \rightarrow \omega_Z(\tilde{D}_Z) \rightarrow \omega_{\tilde{D}_Z} \rightarrow 0, \quad 0 \rightarrow \omega_Y \rightarrow \omega_Y(\tilde{D}_Y) \rightarrow \omega_{\tilde{D}_Y} \rightarrow 0.$$

Push forward the first sequence by f . Since $\text{char } k = 0$ and f is birational between smooth varieties, Grauert–Riemenschneider gives

$$R^i f_*\omega_Z = 0 \quad (i > 0), \quad f_*\omega_Z = \omega_Y.$$

Also $f|_{\tilde{D}_Z} : \tilde{D}_Z \rightarrow \tilde{D}_Y$ is birational between smooth varieties, hence

$$(f|_{\tilde{D}_Z})_*\omega_{\tilde{D}_Z} = \omega_{\tilde{D}_Y}.$$

Therefore

$$0 \rightarrow \omega_Y \rightarrow f_*\omega_Z(\tilde{D}_Z) \rightarrow \omega_{\tilde{D}_Y} \rightarrow 0.$$

²Meaning all its coefficients are 1.

There is a natural map

$$f_*\omega_Z(\tilde{D}_Z) \longrightarrow \omega_Y(\tilde{D}_Y),$$

since the two sheaves agree over the locus where f is an isomorphism. It fits into a commutative diagram with the two exact sequences above, and the left and right vertical arrows are isomorphisms. Hence the middle arrow is an isomorphism:

$$f_*\omega_Z(\tilde{D}_Z) \cong \omega_Y(\tilde{D}_Y).$$

Applying this to f_1 and f_2 yields

$$(\pi_1)_*\omega_{Y_1}(\tilde{D}_1) = (\pi_1 f_1)_*\omega_Z(\tilde{D}_Z) = (\pi_2 f_2)_*\omega_Z(\tilde{D}_Z) = (\pi_2)_*\omega_{Y_2}(\tilde{D}_2).$$

Thus $\pi_*\omega_Y(\tilde{D}) \subseteq \omega_X(D)$ is independent of the choice of embedded resolution. (These are equal as subsheaves of $\omega_X(D)$, since they agree with $\omega_X(D)$ on the open set where (X, D) is already snc.) \square

What do you think?

We won't have the tools to do this next one until next week (March 30th). First a definition.

Definition. Suppose $\pi : Y \rightarrow X$ is a proper birational map between normal varieties and consider a log \mathbb{Q} -Gorenstein pair (X, Δ) . We can always write

$$K_Y - \pi^*(K_X + \Delta) = \sum_{i=1}^t a_i E_i$$

for some prime divisors E_i on Y . We say that a closed subvariety $Z \subseteq X$ is a *non-KLT-center* of (X, Δ) if there exists a Y and $E_i \subseteq Y$ as above such that $Z = \pi(E_i)$ and such that the associated $a_i \leq -1$.

10.

Now suppose that $\phi : F_*^e \mathcal{O}_X \rightarrow \mathcal{O}_X$ is a nonzero map and set $\Delta = \Delta_\phi$. Suppose that $Z \subset X$ is a non-KLT-center of (X, Δ) . Prove that the Z is compatible with ϕ .