Math 7800 Worksheet #4

Due February 27th in gradescope

1.

Let R be a domain finitely generated over k. Fix a Noether Normalization $A = k[x_1, \dots, x_n] \subseteq R$. Now consider a subfield $k' \subseteq k$ such that $[k : k']$ is finite, and the polynomial ring $A' = k'[x_1, \dots, x_n] \subseteq R$. Notice that $A' \subseteq R$ is a Noether normalization of R with respect to k'. Show that there is an isomorphism of R-modules $\mathrm{Hom}_A(R, A) \cong \mathrm{Hom}_{A'}(R, A')$.

Hint: Adjunction of tensor and Hom.

2.

Suppose that X is a variety of finite type over a field k. Show that X has a dualizing complex $\omega_X^\bullet$ so that for every closed point $x \in X$, we have that $(\omega_X^\bullet)_x$ is a normalized dualizing complex for $\mathcal{O}_{X,x}$.

3.

Suppose $(R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$ is a finite local extension of Noetherian local rings. Show that the depth of S as R-module is the same as the depth of S as an S-module. Conclude that S is S_n as an S-module[1] if and only if it is as an R-module.

[1] Recall an R-module M is S_n if for all primes $Q \in \operatorname{Spec} R$ we have that $\operatorname{depth}(M_Q) \leq \min(\dim R_Q, n)$. (Some sources replace $\dim R_Q$ with $\dim M_Q$)

4.

Suppose X is a separated equidimensional d-dimensional scheme finite type of field k. Suppose further that $X = Y \cup Z$ where Y and Z are distinct irreducible components of X. Let $c = \dim Y \cap Z$ and suppose $c < d - 1$.

1. If Y and Z are Cohen-Macaulay, prove that $\mathcal{H}^{-c-1}\omega_X^\bullet \neq 0$.
2. For a greater challenge, if Y and Z are reduced, prove also that $\mathcal{H}^{-c-1}\omega_X^\bullet \neq 0$. You might find the following fact useful, noting that reduced rings are S1.

Theorem. *Suppose $(R, \mathfrak{m})$ is a d-dimensional equidimensional Noetherian local ring with a normalized dualizing complex $\omega_R^\bullet$. We have that $(R, \mathfrak{m})$ is S_n for some $n \leq d$ if and only if*

$$\dim \operatorname{Supp} \mathcal{H}^{-d+i}\omega_R^\bullet \leq d - i - n$$

for every $i > 0$. Here we interpret the dimension of the empty set to be $-\infty$.

5.

Let S be the power series ring $k[[x, y, z]]$ over a field k. For $I = (x, y) \cap (z) = (xz, yz)$, set $R = S/I$, and consider its dualizing complex $\omega_R^\bullet = \mathbf{R}\mathrm{Hom}_S(R, S[\dim S])$. Investigate what happens if we define a "canonical module" $\omega_R = \mathcal{H}^{-\dim R}\omega_R^\bullet$. For example, show that this module is non-zero at one minimal prime of R and is zero at the other. Explain why this implies that the "canonical module" doesn't behave well under localization.

6.

Suppose that $(R, \mathfrak{m})$ is a complete Noetherian F-finite local ring and $I \subseteq R$ is an ideal. Then I is uniformly F-compatible[2] if and only if the image of

$$E_{R/I} \subseteq E \longrightarrow E \otimes_R F^e_* R$$

is annihilated by $F^e_* I$.

[2]Recall this means that $\phi(F^e_* I) \subseteq I$ for all $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ and all $e \geq 0$

7.

Suppose $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring and $J \subsetneq R$ is a proper ideal with $J \cong \omega_R$. Suppose that R/J is F-split (respectively strongly F-regular), prove that then R is also. For generalizations see work of Ma.

Hint: Dualize the sequence $0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$, use that to observe that R/J is Gorenstein. Note a local ring is Gorenstein if its dualizing complex looks like a shift of the ring itself. Additionally note that a map $E \longrightarrow F_* E$ is injective if and only if it is injective on the socle.

8.

Suppose $(R, \mathfrak{m})$ is an F-injective Noetherian local ring that is not necessarily F-finite, with a normalized dualizing complex $\omega_R^\bullet$. Suppose $Q \in \operatorname{Spec} R$. Prove that R_Q is also F-injective. One can find a more general result (without even assuming R has a dualizing complex) in work of Datta-Murayama.

Here's ChatGPT's answer (Pro mode, thinking for 25 minutes, plus about 5 minutes to turn it into LaTeX and shorten it, I didn't give it the hint though).

Proof. Let $(R, \mathfrak{m})$ be Noetherian local of characteristic $p > 0$ with normalized dualizing complex $\omega_R^\bullet$, and assume R is F-injective. Fix $Q \in \operatorname{Spec} R$ and set $S = R \setminus Q$.

Step 1 (duality reformulation). Let $E = E_R(k)$ and $D(-) = \operatorname{Hom}_R(-, E)$. By local duality for a normalized dualizing complex,

$$D\big(\mathbf{R}\Gamma_{\mathfrak{m}}(R)\big) \simeq \omega_R^\bullet.$$

Apply $\mathbf{R}\Gamma_{\mathfrak{m}}(-)$ to Frobenius $F : R \longrightarrow F_*R$ and then D. Using F_* exact and commuting with $\mathbf{R}\Gamma_{\mathfrak{m}}$, and tensor–Hom adjunction, we obtain a natural morphism in $D(R)$

$$\mathrm{T}_F : \ \mathbf{R}\operatorname{Hom}_R(F_*R, \omega_R^\bullet) \longrightarrow \omega_R^\bullet,$$

whose cohomology maps $h^{-i}(\mathrm{T}_F)$ are Matlis dual to $H^i_{\mathfrak{m}}(R) \xrightarrow{F} F_*H^i_{\mathfrak{m}}(R)$. Since $H^i_{\mathfrak{m}}(R)$ is Artinian, Matlis duality is faithful on it, hence

$$R \text{ is } F\text{-injective} \iff h^{-i}(\mathrm{T}_F) \text{ is surjective for all } i. \tag{$*$}$$

Step 2 (localize the trace). Cartier-linear maps localize: for any R-module M,

$$\Big(\operatorname{Hom}_R(F_*R, M)\Big)_Q \cong \operatorname{Hom}_{R_Q}(F_*R_Q, M_Q),$$

via p^{-1}-linear maps $\psi : R \longrightarrow M$ and $\psi_Q(a/s) = \psi(as^{p-1})/s$. Applying this termwise to an injective model of $\omega_R^\bullet$ yields

$$\Big(\mathbf{R}\operatorname{Hom}_R(F_*R, \omega_R^\bullet)\Big)_Q \simeq \mathbf{R}\operatorname{Hom}_{R_Q}(F_*R_Q, (\omega_R^\bullet)_Q),$$

and T_F localizes to the corresponding trace map for R_Q:

$$(\mathrm{T}_F)_Q : \ \mathbf{R}\operatorname{Hom}_{R_Q}(F_*R_Q, (\omega_R^\bullet)_Q) \longrightarrow (\omega_R^\bullet)_Q.$$

Localizing the surjections in $(*)$ gives that $h^{-i}((\mathrm{T}_F)_Q)$ is surjective for all i.

Step 3 (conclude). Up to a shift, $(\omega_R^\bullet)_Q$ is a (normalized) dualizing complex for R_Q, so applying $(*)$ to (R_Q, QR_Q) shows Frobenius acts injectively on $H^i_{QR_Q}(R_Q)$ for all i. Hence R_Q is F-injective. □

Is it right? If not, find the error(s) and fix it, or write your own proof. Here's a hint that I didn't give ChatGPT. *Hint:* Show that R is F-injective if and only if $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(S)$ injects for every *finite* generically purely inseparable ring extension $R \subseteq S$.

9.

Suppose R is a Noetherian F-finite reduced ring with canonical module ω_R. Prove that $T(F_*\tau(\omega_R)) = \tau(\omega_R)$. *Hint:* Note the containment $\subseteq$ is definitional.

10.

Let k be an algebraically closed field of characteristic $p > 0$ and consider the local ring $R = (k[x,y]/(y^2-x^3-x^2))_{(x,y)}$ with maximal ideal $\mathfrak{m} = (x,y)$. It is an integral domain but its completion $\widehat{R}$ is not (see for instance Chapter I, Example 5.6.3 in Hartshorne). Find a submodule of $H^1_{\mathfrak{m}}(R)$ that is F-compatible but does not correspond to a T-compatible submodule of $\omega_R \cong R$.