Math 7800 Worksheet #3

Due February 13th in gradescope

1.

Let $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ and $J \subsetneq R$ be a compatible pair. and let $\overline{\phi}$ be the induced map $F^e_* R/J \xrightarrow{\phi} R/J$. Show that

1. If ϕ is surjective, then so is $\overline{\phi}$.
2. If R is local and $\overline{\phi}$ is surjective, then so is ϕ.
3. Give an example where $\overline{\phi}$ is surjective but ϕ is not (in a non-local setting).

2.

Let S be an F-finite *regular* ring, and R any quotient. Prove that *every* map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ lifts to some map in $\mathrm{Hom}_S(F^e_* S, S)$–that is, there exist $\psi \in \mathrm{Hom}_S(F^e_* S, S)$ compatible with the kernel of $S \twoheadrightarrow R$ and inducing ϕ.

Hint: Kunz's Theorem implies that $F^e_* S$ is a projective S-module.

3.

Show that if $\phi \in \operatorname{Hom}(F^e_* R, R)$ is surjective, then every ϕ-compatible ideal J is radical and $\phi(F^e_* J) = J$.

I asked Utah's ChatGPT 5.2 to solve this on 2/1/2026. Here's what it produced? Is it right, if it is, say so, if it's not, correct it.

Let R be a ring of characteristic $p > 0$ and let $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ be surjective. An ideal $J \subseteq R$ is called ϕ-compatible if $\phi(F^e_* J) \subseteq J$.

Claim 1. If J is ϕ-compatible, then $\phi(F^e_* J) = J$.

Proof. The inclusion $\phi(F^e_* J) \subseteq J$ is clear. Let $x \in J$. Since ϕ is surjective, choose $c \in R$ such that $\phi(F^e_* c) = 1$. Then

$$x = x \cdot 1 = x\,\phi(F^e_* c) = \phi(F^e_*(cx^{p^e})).$$

Since $x^{p^e} \in J$ and J is an ideal, we have $cx^{p^e} \in J$, so $x \in \phi(F^e_* J)$. □

Claim 2. Every ϕ-compatible ideal J is radical.

Proof. Suppose $x^{p^e} \in J$. Using surjectivity, choose $c \in R$ with $\phi(F^e_* c) = 1$. Then

$$x = \phi(F^e_*(cx^{p^e})) \in \phi(F^e_* J) \subseteq J.$$

Thus $x \in J$, so J is radical. □

4.

Prove that if $B \hookrightarrow R$ is a finite extension of Noetherian domains, then there exists $\phi \in \operatorname{Hom}_B(R, B)$ such that $\phi(1_R) \neq 0$.

5.

Suppose R is a Noetherian F-finite domain and $R \subseteq S$ is a finite extension of rings. Prove that

$$\tau(R) \subseteq \operatorname{Image}(\operatorname{Hom}_R(S, R) \xrightarrow{\text{eval at } 1_S} R).$$

Hint: Localize to show that $0 \neq \operatorname{Image}(\operatorname{Hom}_R(S, R) \xrightarrow{\text{eval at } 1_S} R)$ and show that the image is uniformly compatible.

6.

Suppose that R is strongly F-regular. Use the previous exercise to prove that every finite ring extension $R \subseteq S$ splits as a map of R-modules. That is, prove that strongly F-regular rings are **splinters.**[1]

[1]It is an open question whether the converse is true for Noetherian F-finite rings. This is known, for example, for $\mathbb{Q}$-Gorenstein rings by work of Anurag Singh, and rings with finitely generated anti-canonical rings by Chiecchio-Enescu-Miller-Schwede (also proven independently by Singh, but not published).

7.

Let R be a Noetherian Frobenius split ring and let $J \subseteq R$ be an arbitrary ideal. Then show that the annihilator, in R, of the local cohomology module $H^i_J(R)$ is radical for all $i \geq 0$.

Hint: If x^2 is in the annihilator, so is x^p. Now use the Frobenius splitting.

8.

Let $\iota : R \to S$ be a finite extension of Noetherian rings with $(R, \mathfrak{m})$ local. Prove that if ι splits, then F-injectivity of S implies F-injectivity of R, and similarly for F-rationality.

Hint: Note we do not assume that ι is local. Furthermore $\mathfrak{m}S$ need not be maximal, even $\sqrt{\mathfrak{m}S}$ need not be maximal.

Below is ChatGPT's attempt (as of 2/1/2026). It is not correct. What is wrong with it? Write a correct solution.

Proof. Let $\iota : R \longrightarrow S$ be a finite extension of Noetherian rings with $(R, \mathfrak{m})$ local, and assume ι splits as an R-module map. Thus there is an R-linear retraction $\pi : S \longrightarrow R$, so

$$S \cong R \oplus T$$

as R-modules.

F-injectivity. Since S is finite over R, for all i we have

$$H^i_{\mathfrak{m}}(S) \cong H^i_{\mathfrak{m}S}(S).$$

Local cohomology is additive, hence

$$H^i_{\mathfrak{m}}(S) \cong H^i_{\mathfrak{m}}(R) \oplus H^i_{\mathfrak{m}}(T).$$

Frobenius acts functorially on local cohomology and respects this splitting. If S is F-injective, then Frobenius is injective on $H^i_{\mathfrak{m}}(S)$, hence also on the direct summand $H^i_{\mathfrak{m}}(R)$. Therefore R is F-injective. □

9.

Suppose that $(R, \mathfrak{m})$ is a Noetherian F-finite equidimensional reduced local ring. Suppose that $c \in R$ is a strong test element. Prove that $cH^i_{\mathfrak{m}}(R) = 0$ for all $i < d = \dim R$. *Hint:* You may assume without proof that there is $b \in R$ a non-zerodivisor such that $bH^i_{\mathfrak{m}}(R) = 0$.