

MATH 7800 WORKSHEET #2
DUE JANUARY 30TH IN GRADESCOPE

1. Show that the non-Frobenius split locus of an F -finite Noetherian normal ring has codimension at least two.

2. Show that the ring $R = \mathbb{F}_p[x, y, z]/(xyz)$ is Frobenius split.

3. Suppose that k is a field of characteristic $p > 0$ and $x \in k$ is an element without a p th root. Let $L = k(x^{1/p})$. Show that if a k -linear map $\phi : F_*k \rightarrow k$ extends to an L -linear map $\phi_L : F_*L \rightarrow L$, then ϕ is the zero map.

I asked University of Utah's ChatGPT 5.2 to answer this on 1-20-2026 and asked it to convert its answer to LaTeX. Was it correct? Can you do better?

Proof. Suppose

$$\phi : F_*k \rightarrow k$$

is a k -linear map which extends to an L -linear map

$$\phi_L : F_*L \rightarrow L.$$

We show that $\phi = 0$.

Step 1: Determine $\phi_L(1)$. For any $a \in k$, k -linearity of ϕ gives

$$\phi(a^p) = a\phi(1).$$

Since ϕ_L extends ϕ and is L -linear, we also have

$$\phi_L(a^p) = a\phi_L(1).$$

Because $\phi_L(a^p) = \phi(a^p)$ for $a \in k$, we conclude that

$$a\phi_L(1) = a\phi(1) \quad \text{for all } a \in k,$$

and hence

$$\phi_L(1) = \phi(1) \in k.$$

Step 2: Evaluate $\phi_L(x)$. Since $t^p = x$, L -linearity of ϕ_L gives

$$\phi_L(x) = \phi_L(t^p) = t\phi_L(1).$$

But $\phi_L(x) = \phi(x) \in k$, while $t \notin k$. Writing

$$\phi_L(1) = \sum_{i=0}^{p-1} c_i t^i \quad (c_i \in k)$$

in the k -basis $\{1, t, \dots, t^{p-1}\}$ of L , we obtain

$$t\phi_L(1) = \sum_{i=0}^{p-1} c_i t^{i+1} = c_0 t + \dots + c_{p-2} t^{p-1} + c_{p-1} t^p.$$

For this to lie in k , all coefficients of t, t^2, \dots, t^{p-1} must vanish, so $c_0 = \dots = c_{p-2} = 0$. Thus $\phi_L(1) = c_{p-1} t^{p-1}$. Since $\phi_L(1) \in k$ and $t^{p-1} \notin k$, we must have $c_{p-1} = 0$, hence

$$\phi_L(1) = \phi(1) = 0.$$

Step 3: Conclude $\phi = 0$. For any $b \in L$, L -linearity gives

$$\phi_L(b^p) = b\phi_L(1) = 0.$$

Restricting to elements $b \in L$ with $b^p \in k$ (in particular $b = t^i$ with $b^p = x^i$), we obtain

$$\phi(x^i) = 0 \quad \text{for } i = 0, 1, \dots, p-1.$$

Since x is not a p th power in k , it can be extended to a p -basis of k over k^p . The above computation shows that ϕ vanishes on a k^p -basis of k , and therefore $\phi = 0$ on all of k . \square

4. Consider the ring $R = \mathbb{F}_p[x, y, z]/(xy - z^2)$ where $p \neq 2$. Show that there is a Frobenius splitting $F_*R \rightarrow R$ that sends the ideal $F_*(x, z)$ to (x, z) . Show that this splitting then induces a Frobenius splitting of $R/(x, z)$.

Hint: It might help to realize that $R \cong \mathbb{F}_p[a^2, b^2, ab] \subseteq \mathbb{F}_p[a, b]$.

5. Let R be a reduced Noetherian ring. Suppose that $\phi : F_*^e R \rightarrow R$ is an R -linear map. Fix I an ideal and consider $J := \Gamma_I(R) \subseteq R$, the ideal of elements annihilated by a power of I . Prove that $\phi(F_*^e J) \subseteq J$. Conclude that if $Q \subseteq R$ is a minimal prime, then $\phi(F_*^e Q) \subseteq Q$. (We will study this condition—called *compatibility* of J and ϕ —in detail later).

6. Consider the Frobenius split ring $R = \mathbb{F}_p[x, y, z]/(xyz)$ from above. Show that R is not eventually Frobenius split along any element in the maximal ideal $\mathfrak{m} = (x, y, z)$. In particular, conclude the local ring $R_{\mathfrak{m}}$ is not eventually Frobenius split along any non-unit.

7. A reduced Noetherian ring R of characteristic $p > 0$ is called *weakly normal* if $x \in \mathcal{K}(R)$ and $x^p \in R$ implies that $x \in R$. Any ring is called *seminormal* if $x \in \mathcal{K}(R)$, and $x^2, x^3 \in R$ implies that $x \in R$. Show that Frobenius split rings are weakly normal, and that weakly normal rings are seminormal.

8. Fix $p > 2$. Consider the injective ring homomorphism $R = \mathbb{F}_p[x] \xrightarrow{x \mapsto y^2} \mathbb{F}_p[y] = S$. For any i in the range $0 \leq i \leq p-1$, let ϕ_i be the R -module map $F_*R \rightarrow R$ sending $F_*x^i \mapsto 1$ and the other basis elements F_*x^j (with $j \neq i$, $0 \leq j \leq p-1$) of the free R module F_*R to zero.

- (1) Show that ϕ_1 extends to an S -module map $F_*S \rightarrow S$.
- (2) Show that ϕ_{p-1} does not extend to an S -module map $F_*S \rightarrow S$.
- (3) More generally, show that ϕ_i extends to an S -module map $F_*S \rightarrow S$ if and only if $i \leq (p-1)/2$.