I know that~~s~~ it's over, I don't need ~~you~~ plus (~~integral~~) closure.

for simplicity

R int. domain, J a f.g. ideal.

$\bar{J}$ is the int. closure of J.

$$x \in \bar{J} \iff x^n + a_1 x^{n-1} + \dots + a_n = 0$$
$$a_i \in J^i$$

$I_1 \subset J_2$
have $\bar{J}_1 = \bar{J}_2$
iff they have same norm. blo
and expand to so
ideal there

Alternately $\bar{J} =$ ~~$\Gamma(Y, \mathcal{O}_Y)$~~

$$\Rightarrow \Gamma(Y, J \cdot \mathcal{O}_Y) \cap R$$

expand contract, we will try another expand / contract

where $J \cdot \mathcal{O}_Y$ is the ideal sheaf you get by expanding J on each chart, and where Y is normalized blowup of J. (or any blowup that factors through that blowup).

A famous thm of Brançon-Skoda (generalized to all regular rings by Lipman-Sathaye) says

If I is n-generated, then

$$\overline{I^{n+k-1}} \subseteq I^k$$

(note $I^k \subseteq \overline{I^k}$)
so this is a linear control on powers vs integral closure of powers

I want to give a new proof of this that also recovers many other closure-type Brançon-Skoda theorems an other variants.

5.1

R not regular BS, doesn't always hold

Closure type variants?

In char $p>0$, (Hochster-Huneke)

$$\overline{I^{n+k-1}} \subseteq I^k R^+ \cap R$$

$$\begin{array}{c} I^k R^+ \cap R \\ \rotatebox{90}{$=$} \\ (I^k)^+ \ (\subseteq I^{k*}) \end{array}$$

(if R splinters $\Rightarrow I^k R^+ \cap R = I^k$; that's how we measure ~~fail~~ splintery

In mixed char, $0,p>0$

Rodriguez-Villalobos - Schwede

$$\overline{I^{n+k-1}} \subseteq I^k R^{+\wedge p} \cap R$$

$$\begin{array}{c} I^k R^{+\wedge p} \cap R \\ \not\subseteq \ \cap| \\ (I^k)^{epF} \end{array}$$

Heitmann

In char 0, versions w/ ultra filters (Schoutens)

ultra Frobenius

Char indep $\overline{I^{n+k-1}} \subseteq I^k B \cap R$ for B suff. functorial BCM algebra

It turns out, you only need to expand and contract from a certain dg algebra, namely

Set $Y \longrightarrow \operatorname{Spec} R$ to be normalized blowup of R, or just blowup of I^{n+k-1}

I want to expand and contract I^k from $\mathbb{R}\Gamma(Y, \mathcal{O}_Y)$

5.2

$$\mathrm{Ker}\left(R \longrightarrow B/IB\right) = IB \cap R$$

use that

Not clear how to do that.

Option #1. Set $(g_1, \dots, g_m)$ a set of gens of I^k.

Consider

ker ~~(R ⟶~~ $\mathrm{Kos}_\bullet(\underline{g}) \otimes R\Gamma(Y, \mathcal{O}_Y)$

Then $\mathrm{Ann}_R(\mathrm{Kos}_\bullet(g) \otimes R\Gamma(Y, \mathcal{O}_Y))$

$= \mathrm{Ann}_R(H_0(\mathrm{Kos}_\bullet(g) \otimes R\Gamma(Y, \mathcal{O}_Y)))$

$= \ker(R \longrightarrow H_0(\mathrm{Kos}_\bullet(g) \otimes R\Gamma(Y, \mathcal{O}_Y)))$

indep of choice of g_i. Call it $(I^k)^{K_Y}$

(Epstein, McDonald, R.G. —)

If $k=1$, one can show that

$\overline{I^{n-1}} \subseteq (I^k)^{K_Y}$ (I might even sketch this)

Alternate option

or ker

~~ker~~ $\mathrm{Ann}_R(R/I^k \otimes^{L} R\Gamma(Y, \mathcal{O}_Y)) = (I^k)^Y$

Computer suggests ok. ($\overline{I^{n+k-1}} \subseteq (I^k)^Y$)

Happy medium

~~If~~ ∃ generalization of Koszul for powers of ideals, use that

(5.3)

$BE_k(\underline{g})$, gives a free res of R/I^k if ~~g~~ I gen. by a reg. $(f_1, \dots, f_n)$ seq.

(also Eagon–Northcott, also $\mathbb{Z}[x_1, \dots, x_n]/(x_1, \dots, x_n)^n$

$$BE_1(\underline{f}) = \mathrm{Kos}_\bullet(\underline{f})$$

Thm (Ma, McDonald, R.G., Schwede)
R Any ring, (domain). $I = (f_1, \ldots, f_n)$
$$\overline{I^{n+k-1}} \subseteq \mathrm{Ann}_R(BE_k(\underline{f}) \otimes R\Gamma(Y, \mathcal{O}_Y))$$
when Y is the blowup of $\overline{I^{n+k-1}}$
(say normalized blowup if R ~~excellent~~ domain).

Cor $\overline{I^{n+k-1}} \subseteq I^k R^+ \cap R$ (char $p>0$)
$\overline{I^{n+k-1}} \subseteq I^k R^{+\wedge p} \cap R$ (mixed char)

Pf/ (Char $p>0$) $R \to R\Gamma(Y, \mathcal{O}_Y) \to R^+$

$$\ker(R \to H_0(BE_k(\underline{f}) \otimes R\Gamma(Y, \mathcal{O}_Y)))$$
$$\downarrow$$
$$\subseteq \ker(R \to H_0(R/I \otimes^L R^+))$$
$$= \ker(R \to R^+/IR^+).$$

Cor
$$\overline{I^{n+k-1}} \subseteq \ker(R \to H_0(R/I \otimes^L H_0(R\Gamma(Y, \mathcal{O}_Y))))$$
$\forall$ res. of sings or even alt

Cor If R is a birational derived splinter (or even just $R \to R\Gamma(Y, \mathcal{O}_Y)$ splits in $D(R)$)
then $\overline{I^{n+k-1}} \subseteq I^k$.

Fact lots of things weaker than ~~splinters~~ ~~F-reg~~ splinter are birational derived splinter in char $p>0$, mixed char.

Ie
F-rational $\Rightarrow$ pseudo rational $\Rightarrow$ birat. der. spl
not equal
not equal
$\Vert$
param. ideals J ~~are~~ $J = JR^+ \cap R$

Give an idea of proof, $k=1$, $BE_\bullet(\underline{f}) = Kos_\bullet(f)$

Consider $Kos_\bullet(\underline{f}; \mathcal{O}_Y)$ $\qquad (f_1, \dots, f_n)\cdot\mathcal{O}_Y \overset{= \mathcal{L}}{}$ a l.b.

$$0 \to \bigoplus_{\binom{n}{n}} \mathcal{O}_Y \longrightarrow \bigoplus_{\binom{n}{n-1}} \cdots \longrightarrow \bigoplus_{\binom{n}{1}} \mathcal{O}_Y \xrightarrow{\cdot f_i} \mathcal{O}_Y \to 0$$

not exact (locally $(f_1, \dots, f_n)\cdot\mathcal{O}_Y$ is principal,

But there is an exact subcomplex

like on a chart, $(f_i)^{n-2} \xrightarrow{\cdot f_i} (f_i)^{n-1}$

$K_\bullet$

$$\begin{tikzcd}
 & \mathcal{O}_Y \arrow[d, phantom, "\parallel"] & & & & \\
0 \arrow[r] & \bigoplus_1 \mathcal{L}^0 \arrow[r] & \bigoplus_n \mathcal{L}^1 \arrow[r, dashed] & \bigoplus_n \mathcal{L}^{n-2} \arrow[r] & \mathcal{L}^{n-1} \arrow[r] & 0 \\
 & & & 0 \arrow[r] & \mathcal{L}^{n-1} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Apply $\mathbb{R}\Gamma(Y, -)$

complex w/ no homology

$$\begin{tikzcd}
\mathbb{R}\Gamma(Y, \mathcal{L}^{n-1}) \arrow[r] & \mathbb{R}\Gamma(Y, K_\bullet) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathbb{R}\Gamma(Y, Kos_\bullet(\underline{f}, \mathcal{O}_Y)) \arrow[d, phantom, "\parallel"] \\
 & 0 & Kos_\bullet(\underline{f}) \otimes \mathbb{R}\Gamma(Y, \mathcal{O}_Y)
\end{tikzcd}$$

this is bigger than $\overline{I^{n-1}}$

~~If R normal $\overline{I^{n}}$~~

$\overline{I^{n-1}} = \Gamma(Y, \mathcal{L}^{n-1}) \cap R \hookrightarrow \Gamma(Y, \mathcal{L}^{n-1}) \to \mathbb{R}\Gamma(Y, \mathcal{L}^{n-1})$

So $\overline{I^{n-1}} \to 0 \in H_0(Kos_\bullet(f) \otimes \mathbb{R}\Gamma(Y, \mathcal{O}_Y))$

Applications to uniform Artin-Rees

Uniform BS $\Leftarrow$ $\forall$ R excellent reduced

finite dim, $\forall\, l \geq 0$ st $\overline{I^{n+l}} \subseteq I^k$.

5/5