Defn (R,m) excellent (e.g. complete, ess. f.t./ $k, \mathbb{Z}, \mathbb{Z}_p, \ldots$) domain

An R-algebra B is called weakly balanced ~~BCM~~ big CM if $\forall$ sop of R are a reg. seq on B and $mB \neq B$.

Defn (R,m) is BCM regular along ~~$\forall$ BCM~~ B if $R \to B$ is pure. $\Rightarrow$ (R is CM, $H^i_m(R) \hookrightarrow H^i_m(B) = 0$)

In char $p>0$, if Gor, this coincides w/ strongly F-reg; F-finite. $\forall B$.

Thm If (R,m) exc. local domain, ~~then~~ Gorenstein, $0 \neq f \in m$. If R/f is BCM-reg along $B/(fB)$ $\Rightarrow$ R is BCM-reg along B.

~~Ie if R/p is BCM reg $\Rightarrow$~~

Pf/ ~~Exercise~~

$$\begin{tikzcd}
H^{d-1}_m(R) \arrow[r] & H^{d-1}_m(R/f) \arrow[r] \arrow[d, hook] & H^d_m(R) \arrow[r, "\cdot f"] \arrow[d] & H^d_m(R) \arrow[d] \\
H^{d-1}_m(B) \arrow[r] & H^{d-1}_m(B/fB) \arrow[r, hook] & H^d_m(B) \arrow[r] & H^d_m(B)
\end{tikzcd}$$

$H^{d-1}_m(B) = 0$

Thm If (R,m) excellent ess f.t./ char 0, ~~Gorenstein~~ if BCM-regular $\forall B \Rightarrow$ R has rational sings.

Converse? ~~Open~~. False w/out Gor; open if you assume Gor

What are some BCM algebras?

In char $p>0$ R^+ is BCM.

In mixed char $(R^+)^{\wedge p}$, $\widehat{R^+}^m$ are BCM.

4.0

Before, we saw how to define F-split/pure and F-reg. sings, ratl sings, fpt, lct, ppt
(char p>0) (char 0) (char p>0) (char 0) (mixed)

Today we want to link these at a moral level.

R char $p>0$

Equational lemma I Suppose $Y \to \operatorname{Spec} R$ proper (ie projective)

$\eta \in H^i(Y, \mathcal{O}_Y)$ for some $i>0$.

$\exists\ Y' \to Y$ finite surjective such that

$\eta \mapsto 0 \in H^i(Y', \mathcal{O}_{Y'})$.

Cor A $\exists\ Y' \to Y$ st $H^i(Y, \mathcal{O}_Y) \to H^i(Y', \mathcal{O}_{Y'})$ ← note, not $=0$

is zero map

(since $H^i(Y, \mathcal{O}_Y)$ is f.g. $/R$

Cor B $\exists\ Y' \to Y$ st $\tau^{\geq 1} R\Gamma(Y, \mathcal{O}_Y) \to \tau^{\geq 1} R\Gamma(Y', \mathcal{O}_{Y'})$

is zero in derived cat.

Equational lemma II $\forall \eta \in H^i_{\mathfrak{m}}(R)$ st $\{\eta, \eta^p, \eta^{p^2}, \dots\}$

generate a f.g. submod of $H^i_J(R)$ ← not Noeth

$\exists\ R \hookrightarrow R'$ finite st $\eta \mapsto 0 \in H^i_J(R') = H^i_{JR'}(R')$

T

Equational lemma II $J = m$, $\Rightarrow$

· ~~R^+ is BCM~~ $H^i_m(R^+) = 0$ $\forall i < \dim R$.

Also ~~Since~~ $H^i_Q(R^+_Q) = 0$ $\forall i < \dim R_Q$.

that implies R^+ is BCM.

Equation lemma I implies

$\forall Y$, $\exists Y' \to Y$ finite st

$\tau^{>0} R\Gamma(Y, \mathcal{O}_Y) \longrightarrow \tau^{>0} R\Gamma(Y', \mathcal{O}_{Y'})$ is zero in $D(R)$.

$$\Rightarrow \quad R\Gamma(Y, \mathcal{O}_Y) \longrightarrow \underset{\substack{\text{a finite} \\ \text{extension of } R.}}{\underset{\parallel}{\Gamma(Y', \mathcal{O}_{Y'})}} \longrightarrow R\Gamma(Y', \mathcal{O}_{Y'}).$$

$\Rightarrow$ $\forall$ proper ~~&~~ surjective $Y \to \operatorname{Spec} R$,

we have a factorization

$$R \longrightarrow R\Gamma(Y, \mathcal{O}_Y) \longrightarrow R^+.$$

Defn R a splinter if $\forall$ $R \subseteq S \subseteq R^+$ finite

$R \to S$ splits as a map of R-mods (ie $R \to R^+$ is pure).

Defn R is a derived splinter if $\forall$

$Y \to \operatorname{Spec} R$ proper surjective,

$R \to R\Gamma(Y, \mathcal{O}_Y)$ splits in $D(R)$

Defn R is a birational derived splinter if

$\forall$ $Y \to \operatorname{Spec} R$ proper birational (ie $\forall$ blowups)

$R \to R\Gamma(Y, \mathcal{O}_Y)$ splits in $D(R)$.

Interpret $H^i(Y, \mathcal{O}_Y)$ via Cech complex. C^{i}_{Y})

We will just prove first one. Others are similar.

$\exists$ a relation $\eta^{p^n} = r_1 \eta^{p^{n-1}} - r_2 \eta^{p^{n-2}} - \dots - r_n \eta$ $\checkmark$ g acts on $H^i(\dots$

set $g(T) = T^{p^n} - r_1 T^{p^{n-1}} - r_2 T^{p^{n-2}} - \dots - r_n T$ so $g(\eta) = 0$ $\in H^i(Y, \mathcal{O}_Y)$

Note, g additive $g(\alpha+\beta) = g(\alpha) + g(\beta)$

Let $\tilde{\eta} \in C^i_Y$ represent $\eta = [\tilde{\eta}]$.

We see $g(\tilde{\eta}) = d(\beta)$

$\beta \in C^{i-1}_{Y_0}$ so $\beta = (\beta_{\underline{j}}, \dots)$. $\beta_{\underline{j}} \in \Gamma(U_{\underline{j}}, \mathcal{O}_Y) = R_{\underline{j}}$

Now, each $g(T) - \beta_{\underline{j}}$ has a root in some finite extension of $R_{\underline{j}}$. So Pick $Y_1 \xrightarrow{f_1} Y$ finite surjective so each on $\Gamma(f_1^{-1}(U_{\underline{j}}), \mathcal{O}_{Y_1})$, $g(T) - \beta_{\underline{j}}$ has a root. So $\beta = g(\gamma)$ on some Y_1.. $\gamma \in C^i_{Y_1}$

Next, since g additive, Čech is just sums of signed ring maps, different Γ

$$g(\tilde{\eta}) = d(\beta) = d(g(\gamma)) = g(d(\gamma))$$

Thus $g(\tilde{\eta} - d(\gamma)) = 0$ (g additive)

Thus each component of $\tilde{\eta} - d(\gamma)$ is integral over R. Set S to be a finite domain extension containing all roots of $g(T)$, Pick $Y' \to Y$, $\to \operatorname{Spec} S$

... $\tilde{\eta} - d(\gamma)$ components.

$\gamma \in \mathcal{C}^i_{Y'}$

Thus $\tilde{n} - d(\gamma)$ is an element of

the Čech-type constant complex $D^\bullet$

$$D^\bullet: \prod_j \Gamma(Y', \mathcal{O}_{Y'}) \to \prod_{j_1 < j_2} \Gamma(Y', \mathcal{O}_{Y'}) \to \cdots \to \Gamma(Y', \mathcal{O}_{Y'}) \to 0$$

$$\cap$$

$$\mathcal{C}^\bullet_{Y'}$$

But $D^\bullet \simeq \Gamma(Y, \mathcal{O}_Y)$,

that is $H^i(D^\bullet) = 0 \ \forall i > 0$. So $[\tilde{n} - d(\gamma)]$

$= 0 \in H^i(D^\bullet)$, and so it goes to zero in

$H^i(\mathcal{C}^\bullet_{Y'})$. As desired.

Defn R a Noeth domain R is a splinter if $R \to S$ splits, R-m-d, for all finite extensions of domains $R \to S$.

Defn R is a derived splinter, if $\forall\ Y \xrightarrow{f} \operatorname{Spec} R$ proper surjective, $R \to R\Gamma(Y, \mathcal{O}_Y)$ splits in derived cat.

Thm If R is SFR, then R is a splinter.

Converse holds if R = Gorenstein.

Pf/ $\operatorname{Im}(\operatorname{Hom}_R(S,R) \xrightarrow{\text{eval@1}} R) = J$. Then $\varphi(J^{1/p}) \subseteq J$ $\forall\ \varphi : R^{1/p} \to R$. (Exercise)

Conveseley, it suffices to show splinter $\Rightarrow$ SFR for (R,m) SFR on punctured spec. There is an ideal $\tau(R)$ $\leftarrow$ not related to truncation $\subseteq R$ unique smallest ideal compat w/ all $\varphi: R^{1/p^e} \to R$, ~~m-pr~~ punctured spec SFR $\Rightarrow$ $\tau(R)$ is m-primary.

Matlis duality aside $R/\tau(R) \twoheadleftarrow R$ corresponds to $N \hookrightarrow E = H^d_m(R)$. N becomes biggest compat. submodule of $H^d_m(R)$ compat w/ F. N is finite length since $R/\tau(R)$ is. So we can kill N by a finite cover:

$$\begin{tikzcd} & H^d_m(R)/N \arrow[r] & H^d_m(S) \\ \text{cover:} & H^d_m(R) \arrow[u] \arrow[r] & H^d_m(S) \arrow[u] \\ & N \arrow[u, phantom, "\subseteq" rotate=90] \arrow[r] & 0 \arrow[u, phantom, "\subseteq" rotate=90] \end{tikzcd}$$

So

$$\begin{tikzcd} \tau(R) \arrow[d] & \mathrm{Hom}(S,R) \arrow[l] \arrow[d, equal] \\ R & \mathrm{Hom}(S,R) \arrow[l] \end{tikzcd}$$

$\star$

(Matlis dual, local dual exercise for $\underline{H^d_m(S) = \mathrm{Hom}(S,R)^\vee}$)

__Cor__ In char $p>0$, splinter = derived splinter. ($\Leftarrow$ ✓)

Pf

$$\begin{tikzcd} & \Gamma(\mathcal{O}_Y) = S \arrow[r] \arrow[d] & S' = \Gamma(\mathcal{O}_{Y'}) \arrow[d] \\ R \arrow[r] & R\Gamma(Y, \mathcal{O}_Y) \arrow[r] \arrow[d] & R\Gamma(Y', \mathcal{O}_{Y'}) \arrow[d] \\ & \tau^{>0} R\Gamma(Y, \mathcal{O}_Y) \arrow[r, "0"] & \tau^{>0} R\Gamma(Y', \mathcal{O}_{Y'}) \end{tikzcd}$$

This implies a factorization $R\Gamma(Y, \mathcal{O}_Y) \to S' \to R\Gamma(Y', \mathcal{O}_{Y'})$

$\star$ __Exercise__

$$\begin{tikzcd} & A^\bullet \arrow[d] \\ D^\bullet \arrow[ur, dashed, "\exists"] \arrow[r] \arrow[dr, "0"'] & B^\bullet \arrow[d] \\ & C^\bullet \arrow[d, "+1"] \\ & {} \end{tikzcd}$$

So $R \to R\Gamma(Y, \mathcal{O}_Y) \to S'$ splits, $\Rightarrow$ first map splits.

Summary

$\underline{\text{Char } p>0}$ SFR $\overset{?\text{ yes (open)}}{\Rightarrow}$ Splinter = Derived splinter

$\underline{\text{Char } 0}$ Rational = Derived splinter

$\Rightarrow$ Splinter = normal.

(essentially shows SFR type $\Rightarrow$ ratl)

What's coming up?

- Explore extension and contraction of ideals along maps $R \to R\Gamma(\mathcal{O}_Y)$. (Application Briançon-Skoda)
- Explore what happens in mixed characteristic.

~~Thm~~ Thresholds. (R, m) local, $0 \neq f \in m$

In char $p>0$. $\mathrm{fpt}(f) = \sup\{t>0 \mid R \to S$ splits, $1 \mapsto f^t$ $\forall S$ w/ $f^t \in S$.

for all $t>0$ (Rodriguez-Villalobos).

$= \sup\{t>0 \mid R \to R\Gamma(Y, \mathcal{O}_Y)$ splits, $1 \mapsto f^t\}$

(other direction is a version of SFR $\Leftrightarrow$ splinter we did.

In char 0

$\mathrm{lct}(f) = \sup\{t>0 \mid R \to R\Gamma(\mathcal{O}_Y)$ splits, $1 \mapsto f^t\}$.

This essentially shows $\mathrm{lct}(f) \geq \mathrm{fpt}(f_p)$ (going mod p)