(A, m) Regular local ring F-finite of char $p>0$. $0 \neq f \in m$.

We know for some $e>0$,

$$\begin{array}{ccc} A & \longrightarrow & A^{1/p^e} \quad \text{splits} \\ 1 & \longmapsto & f^{1/p^e} \end{array}$$

Also works for $f^2, f^3, \ldots$

Natural defn. F-pure threshold (fpt)

$$\mathrm{fpt}(f \in A) = \sup\left\{ \frac{a}{p^e} \;\middle|\; \begin{array}{ccc} A & \longrightarrow & A^{1/p^e} \\ 1 & \longmapsto & f^{a/p^e} \end{array} \text{ splits} \right\}$$

(a limit) $= \sup\left\{ \frac{a}{p^e} \;\middle|\; f^{a/p^e} \notin m A^{1/p^e} \right\}$

Exercise $\mathrm{fpt}(f \in A) \leq 1$.

Rough idea. larger $\mathrm{fpt}(f) \Rightarrow$ less singular $R = A/(f)$.

Examples $\mathrm{fpt}(xy) = 1$

$\mathrm{fpt}(x^2 - yz) = 1$

$\mathrm{fpt}(xy(x-y)) = 2/3$ if $p \equiv_3 1$ (other p exercise)

$\mathrm{fpt}(y^2 - x^3) = 5/6$ if $p \equiv_6 1$ (do more in a sec)

Lemma $R = A/(f)$ is F-split $\Leftrightarrow \mathrm{fpt}(f) = 1$

$(\Leftarrow)$

$$\begin{array}{ccc} A & \longrightarrow & A^{1/p} \quad \text{splits (Choose } a/p^e \geq \frac{p-1}{p}) \\ 1 & \longmapsto & f^{p-1/p} \end{array}$$

$$\begin{array}{ccccc} A & \longrightarrow & A^{1/p} & \longrightarrow & A^{1/p^e} \\ 1 & \longmapsto & f^{p-1/p} & \longmapsto & f^{a/p^e} \end{array}$$

3.1

$$\begin{tikzcd} A \arrow[r] & A^{1/p} \arrow[r] \arrow[rr, bend left, "\varphi"] & A^{1/p} \arrow[r, "\psi"] & A \\ 1 \arrow[r, mapsto] & 1 \arrow[r, mapsto] & f^{\frac{p-1}{p}} \arrow[r, mapsto] & 1 \end{tikzcd}$$

Then $\varphi((f^{1/p})) = \psi(f^{\frac{p-1}{p}}(f^{1/p})A^{1/p})$

$= \psi(fA^{1/p}) = f\psi(A) = (fA)$

So $\psi((f^{1/p})A^{1/p}) \subseteq fA$

$$\begin{tikzcd} (f^{1/p}) \arrow[d, hook] & & (f) \arrow[dl, hook'] \\ A^{1/p} \arrow[r, "\psi"] \arrow[d] & A \arrow[d] & \\ A^{1/p}/(f^{1/p}) \arrow[r, two heads] & A/(f) & \end{tikzcd}$$

$1 \mapsto 1$ surjective, $\Rightarrow A/(f)$ is F-split

Example: $f = y^2 - x^3 \in k[x,y]$

$$\mathrm{fpt}(f) = \begin{cases} 5/6 & p \equiv_6 1 \\ 1/2 & p \equiv_6 2 \\ 2/3 & p \equiv_6 3 \\ \frac{5}{6} - \frac{1}{6p} & p \equiv_6 5 \end{cases}$$

$\sup\{t \in \mathbb{Q}_{\geq 0} \mid \forall S \supseteq R, f^t \in S$, $R \to S$ splits$\}$
$1 \mapsto f^t$

Recall R^+

~~Other formulations~~ Other formulation.

Rodriguez-Villalobos.

$\mathrm{fpt}(f) = \sup\{t \in \mathbb{Q}_{\geq 0} \mid R \longrightarrow R^+$ pure$\}$ units
$1 \mapsto f^t$

3.2

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$= \sup\{t \in \mathbb{Q}_{\geq 0} \mid f^t \notin mR^+\}$

LCTs (A, m) regular ess. f.t. $/k = \cancel{\mathbb{F}_p}$ char $k=0$.

$0 \neq f \in m$.

LCT = log canonical threshold

$lct(f \in A) = \sup\{t \in \mathbb{Q}_{>0} \mid \exists R \subseteq S$ finite normal domain, $f^t \in S$ and $Y \to \operatorname{Spec} S$ a RoS $R \to S \to R\Gamma(Y, \mathcal{O}_Y)$ splits$\}$

$1 \mapsto f^t$

Examples $lct(xy \in k[x,y]) = 1$

$lct(xy(x-y)) = 2/3$

$lct(y^2 - x^3) = 5/6$

Facts $0 < lct(f) \leq 1$.

$lct(f) = 1 \iff A/(f)$ is log canonical (didn't define, but take it as defn).

~~Fact~~ Thm (Takagi–Watanabe, cf Smith, Hara, Mehta–Srinivas)

$A_{\mathbb{Z}}$ f.t.$/\mathbb{Z}$, $J_{\mathbb{Z}} \subseteq A_{\mathbb{Z}}$ is an ideal

$J_{\mathbb{Q}} =$ st $J_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq A_{\mathbb{Q}}$ $= A \otimes \mathbb{Q}$ is maxl

$A = \mathbb{Z}[x_1, \dots, x_n]$, $m_A = (x_1, \dots, x_n)$, $f \in m_A$

$f \in A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$, $m_{\mathbb{Q}} = m_A \otimes \mathbb{Q}$. $\bar{f} \in A_p$, $m_p = m_A \otimes_{\mathbb{Z}} \mathbb{F}_p$

$$lct(f \in A_{\mathbb{Q}, m_{\mathbb{Q}}}) = \lim_{p \to \infty} fpt(\bar{f} \in A_p)$$ 3.3

What to do about mixed char?

(A,m) char 0 ring but char $A/m = p > 0$

ie $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p[[x_2, \dots, x_n]]$.

$0 \neq f \in m$. What is an analog of ~~fpt~~, lct, etc.
What are singular f's $(f \in m^2)$ ($x^4 + y^4 + z^4 \in \mathbb{Z}_p[x,y,z]$)
($p^4 + y^4 + z^4 \in \mathbb{Z}_p[[y,z]]$)

In char 0, $R\Gamma(Y, \mathcal{O}_Y)$ (for a res. of sings of $S \supseteq R$ is CM)

In char $p>0$, $R_{perf} = \bigcup R^{1/p^e}$ is almost CM
and we will see tommorow R^+ is ~~CM~~ BCM
$(H^i_m(R^+) = 0 \ \forall i < \dim R)$.

In mixed char, neither works, but

Thm (Bhatt) (R,m) mixed char excellent local domain. Then $R^{+\wedge p}$ is ~~CM~~ BCM

(A,m) regular, complete

Defn $\mathrm{ppt}(f) = \sup\{t \in \mathbb{Q}_{>0} \mid \cancel{A} \longrightarrow \cancel{A}^{+\wedge p}$, $f \in A$
$1 \longmapsto f^t$
is pure $\}$

$= \sup\{t \in \mathbb{Q}_{\geq 0} \mid \cancel{A}\ f^t \notin mA^{+\wedge p}\}$

$= \sup\{t \in \mathbb{Q}_{\geq 0} \mid f^t \notin mA^+\}$ (3.4)

$\sup\{t \in \mathbb{Q}_{\geq 0} \mid \cancel{A} \ \forall S \supseteq A$ finite, $f^t \notin \cancel{A}, A \to S$ splits$\}$ $= \sup\{t \in \mathbb{Q}_{\geq 0} \mid A \longrightarrow A^+$ pure $\}$
$1 \longmapsto f^t$

So what do we know?

(A, m) regular local mixed char $(0, p>0)$
A/ϖ also regular but of char $p>0$.

$ppt(f \in A) \geq fpt(\bar{f} \in A/\varpi)$

$ppt(f \in A) \leq lct(f \in A)$ defined over all blowups.

Sometimes these agree.

$f = y^2 - x^3 \in \mathbb{Z}_p[\![x,y]\!]$, $p \equiv_6 1$ $fpt(\bar{f}) = 5/6$
$lct(f \in A) = 5/6$

Somes you can also do this w/ associated graded.

Other times not
Cai - Panda - Quinlan - Gallego, Tucker
Benozzo, - Jagathese - Pandey, - Ramirez - Moreno
Sridhar

Yoshikawa,
Yamaguchi

~~Vast~~ $f = p^3 + x^2 + y^3$ in $\mathbb{Z}_2[\![x, y]\!]$

$fpt(z^3 + x^3 + y^3) = \frac{1}{2} < ppf(f) \leq 3/4 < 1 = lct$

3.5

Various examples are not known Tucker's Bane $x^3 + 27 \in \mathbb{Z}_3[\![x]\!]$