(R, m, k) Noeth ~~local~~ domain ~~ring~~ / ess. f.t. over a field k of char 0. Obviously no Frobenius morphism.

How do we measure singularities?

Various tools, D-modules, resolution of sings, ~~etc.~~ topological approaches (for $k = \mathbb{C}$). We use RoS (res. of sings)

What is that?

$\pi : Y \to \operatorname{Spec} R$
- proper (always assume projective) → may as well assume blowup
- birational (implies surjective)
- Y is nonsingular ($\mathcal{O}_{Y,y}$ regular local ring)

Exists by Hironaka

Additionally, we can assume (0) Blowup – (nonsing centers)

① $\pi : Y \to \operatorname{Spec} R$ is an iso on (open) locus where $\operatorname{Spec} R$ is nonsingular.

② $\operatorname{Exc}_{\pi} = E$ is a SNC divisor on Y

$\|$ locus where π not an iso

each irred. comp of E nonsing. and

SNC means at each $y \in E \subseteq Y$, at $\mathcal{O}_{Y,y} = (S, x, \ldots$

$\mathcal{O}_{Y,y} \to \mathcal{O}_{E,y}$ ker is $(x_1^{a_1} \cdots x_d^{a_d})$ $(0 \le a_i \in \mathbb{Z})$

③ For any ideal $J \subseteq R$, can assume $V(J\mathcal{O}_Y) + \operatorname{Exc}$ is a SNC divisor

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Examples $R = k[x,y,z]/(x^4+y^4+z^4)$

Blowup origin (x,y,z)

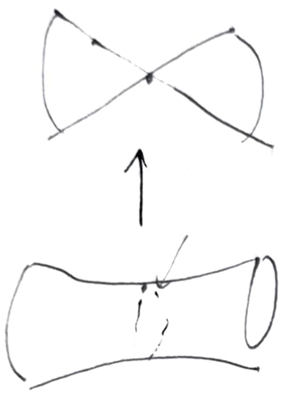
Three charts $\operatorname{Spec}\left(k\left[x, \frac{y}{x}, \frac{z}{x}\right] \Big/ \left(1 + \left(\frac{y}{x}\right)^4 + \left(\frac{z}{x}\right)^4\right)\right)$

only need 2 charts

Original sing. at origin. $\left(\frac{\partial f}{\partial x}=4x^3, \frac{\partial f}{\partial y}=4y^3, \frac{\partial f}{\partial z}=4z^3, f\right)$ ← m-primary ideal

New chart$_{\text{Spec}}$ $k[x,y',z']/(1+y'+z')$ ←

$\left(\frac{\partial f'}{\partial x}=0, \frac{\partial f'}{\partial y'}=4y'^3, \frac{\partial f'}{\partial z'}=4z'^3, f'\right)$ Vanishing locus is $\emptyset$



resolution of a cone sing.

$E = V(x)$ on our chart

It satisfies ①, ②, $J=(x,y,z)$ works since $E = V(J\cdot\mathcal{O}_Y)$

Exercise Let $R=k[x,y]$, $J=(y^2-x^3)$, blowup origin to find a R.o.S. satisfying ③ (3 blowups)

How do you use to measure sings?

One way use $\mathcal{O}_Y$ as a replacement for something like Frobenius, but how? $\Gamma(Y,\mathcal{O}_Y)=R^N$ = normalization of R.

Doesn't say much. Alternately $\mathbb{R}\Gamma(Y,\mathcal{O}_Y)$, complex that computes $H^i(Y,\mathcal{O}_Y)$

- Take an injective res. of $\mathcal{O}_Y$, apply $\Gamma(Y,-)$ functor $\longrightarrow \mathbb{R}\Gamma(Y,\mathcal{O}_Y)$
- Use Čech description.

Set $U_1,\dots,U_n$ open affine cover, $U_i\cap U_j$ etc. affine

$$\check{C}^\bullet(\{U_i\}) = \prod_i \Gamma(U_i,\mathcal{O}_Y) \to \prod_{i<j}\Gamma(U_i\cap U_j,\mathcal{O}_Y) \to \cdots \to \Gamma(\textstyle\bigcap_i U_i, \mathcal{O}_Y)$$

Example $\Gamma(U_1,\mathcal{O}_Y)\oplus\Gamma(U_2,\mathcal{O}_Y)\to\Gamma(U_1\cap U_2,\mathcal{O}_Y)$

$$0\to k[x,\tfrac{y}{x},\tfrac{z}{x}] \oplus k[\tfrac{x}{y},y,\tfrac{z}{y}] \to k[x,\tfrac{y}{x},\tfrac{z}{x},\tfrac{x}{y}] \to$$

(2.

Exercise Compute $H^1(Y, \mathcal{O}_Y)$ for our example
$$R = k[x,y,z]/(x^4+y^4+z^4)$$

Exercise M2 computation via BGG package

Defn R has rational sings if $R\Gamma(Y, \mathcal{O}_Y) \cong R$ (iso in der. cat) for some (equivalently all) resolution of sings, ~~then~~

Example $R = k[x,y,z]/(x^4+y^4+z^4)$ does not have rational sings since $H^1(Y, \mathcal{O}_Y) \neq 0$ (as you will show).

Example $R = k[x,y,z]/(xy - z^2)$ does have ratl sings.

Rational singularities are nice

1. Rational sings are normal, $R \cong \Gamma(Y, \mathcal{O}_Y) \cong R^N$.
2. Rational sings are CM.

Thm (Grauert–Riemenschneider vanishing). $m \in R$ maxl

$$H^i_m(R\Gamma(Y, \mathcal{O}_Y)) = 0 \quad \forall i < \dim R \quad (\text{all } i \neq \dim R)$$

[What do I mean $R\Gamma(Y, \mathcal{O}_Y)$ is iso. to a bd below complex of injectives $I^\bullet$, $H^i_m(R\Gamma(Y, \mathcal{O}_Y)) = H^i(\Gamma_m(I^\bullet))$]

So $H^i_M(R) = H^i_m(R\Gamma(Y, \mathcal{O}_Y)) = 0 \quad \forall i < d, \Rightarrow R$ is CM.

Aside on trace.

$R \subseteq S$ finite extension of Noeth domains, R normal.

$K(R) \subseteq K(S)$ a finite extension of fields.

$$K(S) \cong S \otimes_R K(R)$$

$\forall y \in K(S)$, we have $\cdot s : K(S) \to K(S)$
an ~~R~~ $K(R)$-linear map, of f.d. $K(R)$-vspaces

We can take trace of $\cdot s$

$s \mapsto$ trace of $(\cdot s)$

This gives an R-linear map $Tr : K(S) \to K(R)$
not hard to show $Tr(s) \in R$ if $s \in S$.

This gives a map $S \to R$ (R-linear).

Facts
- $Tr : K(S) \to K(R)$ is zero $\iff$ $K(R) \subsetneq K(S)$ inseparable.
- $Tr(1_S) = \cancel{\deg}\ [K(S):K(R)]$. ⟵ exercise
- In char 0, $Tr : S \to R$ is surjective,
- $\frac{1}{\deg} Tr$ splits every finite extension.

Thm (Kovács, Bhatt) (converse ok too)

Suppose X a variety w/ rtl sings, and $Z \xrightarrow{\pi} X$ proper surjective, then $\mathcal{O}_X \to R\pi_*\mathcal{O}_Z$ splits. (in der cat.) In particular, $\forall R \subseteq S$, $\forall Y \to \operatorname{Spec} R$ a res. of sings $R \to S \to R\Gamma(Y, \mathcal{O}_Y)$ splits.

Pf/ Idea. $\mathcal{O}_X \to R\pi_*\mathcal{O}_Z \to R\pi_*\mathcal{O}_{Z'} \to R\pi_*\mathcal{O}_H$

hyperplane sections

projective

$$\begin{array}{ccc} & \text{normal} & \text{finite} \\ & Y & \xleftarrow{\mu} Z \\ \text{birat} & \downarrow \pi & \downarrow \\ & & H \\ & \swarrow & \downarrow \text{biratl} \\ & X & \xleftarrow{\text{finite}} X' \end{array}$$

$\mathcal{O}_X = \cancel{R\Gamma(Y,\mathcal{O}_Y)}\ R\pi_*\mathcal{O}_Y$

$\mathcal{O}_Y \to R\mu_*\mathcal{O}_Z$ splits by trace

so $\mathcal{O}_X \xrightarrow{\sim} R\pi_*\mathcal{O}_Y \underset{\text{splits by trace}}{\longrightarrow} R\mu_* R\pi_* \mathcal{O}_Z$

splits.

~~Cor~~ R has ratl sngs iff $\forall\ Y \to \operatorname{Spec} R$ surj. (alteration) $R \to R\Gamma(\mathcal{O}_Y)$ splits.

<u>Defn</u> lct. (log canonical threshold) $f \in R$, R Gorenstein (re regular)

Suppose $f^{\frac{1}{b}} \in S$, $Y \to \operatorname{Spec} S$ a resolution of sngs

Consider $R \to S \to R\Gamma(Y, \mathcal{O}_Y)$.

$1 \longmapsto f^{\frac{a}{b}}$

$$\operatorname{lct}(R,f) = \sup\left\{\tfrac{a}{b} \;\middle|\; \underbrace{R \xrightarrow{1 \mapsto f^{a/b}} S \to R\Gamma(Y,\mathcal{O}_Y)}_{\text{splits}}\right\}.$$

In fact if $Y \to \operatorname{Spec} R$ is a resolution of sngs, satisfying 3 (for $J = (f)$) enough to check that $R \to R\Gamma(Y, \mathcal{O}_Y(\lfloor \tfrac{a}{b} \operatorname{div}_Y(f) \rfloor))$ splits.

<u>Thm</u> R Gor, char. 0, $R = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ $\quad R_{\mathbb{Z}}$ f.t./$\mathbb{Z}$.

$R_p = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p$.

<u>TFAE</u> ① R has rat sngs

② R_p is SFR $\forall\ p \gg 0$

(can even ...)

$f \in R_{\mathbb{Z}}$, $f_p = \operatorname{im} \in R_p$

$\lim_{p \to \infty} \operatorname{fpt}(f) = \operatorname{lct}(f)$

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