(R, m) A Noeth local ring (domain)

(little lost if $R = \hat{R}$, or R ess. f.t. over k or $\mathbb{Z}$).

How do we study sings of R?

We will take perspective of studying

$R \hookrightarrow R'$ (usually injective) or $X \to \operatorname{Spec} R$ (usually surjective)

or $A \twoheadrightarrow R$ (A nonsingular, regular) and understand
$\quad\quad\quad\quad\;\; {}^{\shortparallel}A/I$

how I behaves under maps $A \to A'$ ($Y \to \operatorname{Spec} A$)

Beginning of story, in characteristic $p > 0$.

R has a special map? (Audience).

$$F : R \longrightarrow R, \quad x \longmapsto x^p$$
That's confusing notationally...

One solution: let $R^{1/p}$ denote the set of pth roots of $R \subseteq \overline{K(R)}$. Then

$$\begin{tikzcd} F: R \arrow[r, "x \mapsto x^p"] \arrow[dr, "x \mapsto ?"'] & R \arrow[d, "\cong"] & y \arrow[d, mapsto] \\ & R^{1/p} & y^{1/p} \end{tikzcd}$$
(+)

So Frobenius is identified w/ inclusion

$R \hookrightarrow R^{1/p}$ $(x \mapsto x)$.

<u>Thm</u> (Kunz, 19) (R,m) Noeth. $R^{1/p}$ is a flat R-module $\Leftrightarrow$ R regular ($\dim R = \#$ gens m).

Example $R = k[x_1, \dots, x_n]_m$. (k is perfect, $k^p = k$ or $k = k^{1/p}$)

$$R^{1/p} = k^{1/p}[x_1^{1/p} \dots x_n^{1/p}]_m = k[x_1^{1/p} \dots x_n^{1/p}]_m \quad \text{a free } R\text{-module}$$

w/ basis $1, x_1^{1/p}, x_2^{1/p}, \dots, x_n^{1/p}, x_1^{1/p}x_2^{1/p}, \dots, x_1^{p-1/p} \dots x_n^{p-1/p}$ (free $\Rightarrow$ flat)

$$= \{x_1^{a_1/p} \dots x_n^{a_n/p}\}_{0 \le a_i < p} \quad \text{(all exponents} \le p-1\text{)}$$

Proof of Kunz ($\Leftarrow$) ~~Proof~~ Option #1. Complete R

$\hat{R} = \varprojlim R/m^e$, then $R \cong k[[x_1, \dots x_n]]$ (pf looks like above after dealing w/ fact that k not perfect).

Option #3 $R \to R^{1/p}$ local map ($m^{1/p} \cap R = m$) of Noeth rings, st

① R is regular (hyp.)

② $\dim(R^{1/p}) = \dim R + \dim(\overset{0}{\overbrace{R^{1/p}/mR^{1/p}}})$

③ $R^{1/p}$ is CM ($H^i_{m^{1/p}}(R^{1/p}) = 0 \ \forall i < \dim R = \dim R^{1/p}$)

Then $R \to R^{1/p}$ is flat. (Miracle flatness).

Option #2 (F-finite)
$\operatorname{pdim}(R^{1/p}) + \operatorname{depth}(R^{1/p}) = \operatorname{depth}(R)$ (A-B)
So $R^{1/p}$ projective

Pf of Kunz ($\Rightarrow$) We skip it.

What are properties of $R \to R^{1/p}$ in general?

① Local

② Integral extension (frequently finite)

ⓐ Surjective on Spec$R$

ⓑ $m \cdot R^{1/p} \ne R^{1/p}$

Assume $R \hookrightarrow R^{1/p}$ is a finite extension, R regular. (F-finite)

Then $R^{1/p}$ finite + flat $\Rightarrow$ free (R local),

So $\exists\ \pi : R^{1/p} \twoheadrightarrow R$ (project onto a factor)

$r^{1/p} \longmapsto 1.$

Consider $\rho : R^{1/p} \xrightarrow{\cdot r^{1/p}} R^{1/p} \longrightarrow R$ (where is $1^{1/p}$ sent?)

Hence if R regular $\Leftarrow$ F-finite then $R \to R^{1/p}$ splits

(defn R is F-split)

Even more Suppose $0 \neq c \in R$, then $c \notin m^{d}$ for some $d \gg 0$, (since $\bigcap m^{\ell} = 0$)

so $c \notin (x_1^{d}, \ldots, x_n^{d})$ for some n $(m = (x_1, \ldots, x_n))$

so $c \notin (x_1^{p^e}, \ldots, x_n^{p^e})$ for some $e \gg 0$.

So $c^{1/p^e} \notin \underbrace{(x_1, \ldots, x_n)R^{1/p^e}}_{= mR^{1/p^e}}$ (here R^{1/p^e} = set of p^eth roots of R in $\overline{K(R)}$, also $\cong R$)

So $\overline{c^{1/p^e}} \in R^{1/p^e}/mR^{1/p^e}$ is part of a basis for $R^{1/p^e}/mR^{1/p^e}$ over R/m $\overset{NAK}{\Rightarrow}$ c^{1/p^e} is part of a basis for $R^{1/p^e}/R$, thus $\exists\ \rho : R^{1/p^e} \xrightarrow{c^{1/p^e} \mapsto 1} R$.

So for every $0 \neq c \in R$, $\exists e > 0$ st $\underset{1 \mapsto c^{1/p^e}}{R \to R^{1/p^e}}$ splits.

Def R is strongly F-regular (SFR).

Examples

- $k[x,y,z]_m/(x^2 - yz)$ is not regular $(x^2 - yz \in m^2 \setminus m)$

  but it is SFR, hence F-split

- $k[x,y,z]_m/(x^3 + y^3 + z^3)$ $(p \neq 3)$ ~~$(p \neq 3)$~~ ~~$(p \equiv_3 ...)$~~

  not ~~F-sf~~ SFR

  - it is F-split if $p \equiv_3 1$
  - it is not F-split if $p \equiv_3 2$

- $\dfrac{k[x,y,z]}{(f)}$ $\deg f \geq 4$, f homog, not F-split.

How do you check this? (later, but first tools)

- To show R F-split suffices to exhibit a splitting
- To show R is <u>not</u> F-split, suffices to show an ideal $I \subseteq R$ st $IR^{1/p} \cap R \supsetneq I$ (<u>lemma</u> exercise if $R \to S$ splits then $IS \cap R = I$ $\forall I \subseteq R$) ☆
  under moderate hyp. $\Leftrightarrow$ $R \to S$ splits
- To show R is ~~not~~ SFR, suffices to show $\begin{matrix} R \to R^{1/p^e} \\ 1 \mapsto c^{1/p^e} \end{matrix}$ splits for some $0 \neq c \in R$ st $R[c^{-1}]$ regular (all loc. are reg.)
- To show R is not SFR, suffices to find $J \subseteq R$ st $\varphi(J^{1/p^e}) \subseteq J$ $\forall$ $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ $\forall e > 0$.
  (sometimes, R Gor, suffices to check $R^{1/p} \xrightarrow{\Phi} R$, $e=1$, one special Φ).

Or <u>Macaulay 2</u> TestIdeals package (exercises) ☆

<u>Variant</u> (A, m) regular local ring, F-finite, char $p > 0$, $f \in m$.
$R = A/(f)$, another way to measure sings of R is to consider Frobenius on A, $\begin{matrix} A \mapsto A^{1/p^e} \\ 1 \mapsto f^{a/p^e} \end{matrix}$

What powers a have $\begin{matrix} A \to A^{1/p^e} \\ 1 \mapsto f^{a/p^e} \end{matrix}$ split

$$\operatorname{fpt}(f) = \sup \left\{ \frac{a}{p^e} \;\middle|\; \begin{matrix} A \to A^{1/p^e} \\ 1 \mapsto f^{a/p^e} \end{matrix} \text{ splits} \right\}.$$

☆ Lemma Exercise
$\operatorname{fpt}(f) \leq 1$.