Exercises, Lecture #5

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I Macaulay2 exercises

Remember, you can run Macaulay2 online using the two links below if you don't have it installed on your computer.

- Macaulay2 at Georgia Tech
- Macaulay2 at University of Melbourne

Go and download the following file.

KHClosure Macaulay2 Package
`https://kschwede.github.io/M2/KHClosure.m2`

- If you are running Macaulay2 locally, put the file in a place Macaulay2 will find it, usually your home directory works. Alternately, start Macaulay2 (or emacs/vscode) from the folder where the file works.
- If you are running Macaulay2 on the web, click on "Upload File" in the upper right. Then select, "Local file upload", and select the `KHClosure.m2` file you downloaded. Click ok, and hopefully it will upload.

Regardless, try running the command:

```
needsPackage "KHClosure"
```

If all went well, you shouldn't get an error.

This package lets you compute

$$I^{\mathrm{KH}_Y} = \ker\left(R \to H_0(\mathrm{Kos}(\underline{f}) \otimes \mathbf{R}\Gamma(Y, \mathcal{O}_Y))\right)$$

where Y is a resolution of singularities (or any blowup). We leave off the subscript Y when Y is a resolution of singularities. The syntax to do that is as in the following example which you should try out.

```
S = QQ[x,y,z]
f = x^4+y^4+z^4
R = S/ideal(f)
m = ideal(x,y,z)
I = ideal(x^3, y^3, x*y*z^2+y^2)
koszulHironakaClosure(I, m)
```

In the above, the ideal `m` is the ideal whose blowup gives a resolution of singularities. Because `R` is a cone over a smooth projective curve, blowing up the homogeneous maximal ideal resolves the singularities. (This will work with any standard graded ring that is regular outside the irrelevant ideal). If you put another ideal there, it will blow that up instead.

You can compute the integral closure of an ideal via the following code.

```
integralClosure(I)
```

1. Verify the theorem in we proved today $(I^n) \subseteq \overline{I}^{\mathrm{KH}}$ in several examples. At least choose several different I.

2. Explore the ring $R = \mathbb{Q}[a, b, c, d]/ideal(a^3 + b^3 + c^3 + d^3)$. Show that $I^{\mathrm{KH}} = I$ for at least 5 different ideals. In this case, the ring has rational singularities, is that what you expect? It's also graded with a singularity only at the origin.

3. Go back to $R = \mathbb{Q}[x, y, z]/ideal(x^4 + y^4 + z^4)$. (or even increase the exponent 4 to something bigger).

   (a) Find the simplest ideal you can where $I^{\mathrm{KH}} \neq I$.

   (b) Verify that $(I^{\mathrm{KH}})^{\mathrm{KH}} = I^{\mathrm{KH}}$ in multiple examples.

4. Consider $R = \mathbb{Z}/5[x, y, z]/ideal(x^4 + y^4 + z^4)$. Find an ideal I and an element $x \in I^+ \notin I$ by showing that

$$x \in \ker\Big(R \to H_0(\mathrm{Kos}(\underline{f}) \otimes \mathbf{R}\Gamma(Y, \mathcal{O}_Y))\Big)$$

   for some blowup Y. (You can use a resolution, it's still a blowup)