

Exercises, Lecture #3 and #4

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I Thresholds of singularities

1. Suppose that R is a Noetherian regular ring of characteristic $p > 0$. Prove that $\text{fpt}(f) \leq 1$.

2. Suppose (A, \mathfrak{m}) is an F -finite regular local ring of dimension d . Recall that

$$\text{fpt}(f) = \sup\left\{\frac{a}{p^e} \mid A \xrightarrow{1 \mapsto f^{a/p^e}} A^{a/p^e}\right\}.$$

Suppose $\mathfrak{m} = (x_1, \dots, x_d)$. Prove that

$$\text{fpt}(f) = \sup\left\{\frac{a}{p^e} \mid f^a \notin (x_1^{p^e}, \dots, x_d^{p^e})\right\}.$$

3. Compute the F -pure threshold (fpt) of $xy(x - y) \in \mathbb{F}_2[x, y]$.

4. Suppose that (R, \mathfrak{m}) is a regular local ring essentially finite type over a field of characteristic zero and $f \in \mathfrak{m}$. Prove that $\text{lt}(f) \leq 1$.

5. Suppose p is a prime and B is a ring such that $p \in B$ is not a unit. Suppose additionally that p is not a zero divisor in B (this isn't necessary, but it makes the following definition easier). We say that B is *perfectoid* if the following to conditions hold.
- (a) There exists an element $\varpi \in B$ such that ϖ^p agrees with p up to a unit, $\varpi^p = up$.
 - (b) B is p -adically complete.
 - (c) The map Frobenius map $B/(p) \rightarrow B/(p)$ has kernel $(\varpi)/(p)$. In other words $B/(\varpi) \xrightarrow{r \mapsto r^p} B/(p)$ is an isomorphism.

Now suppose that (R, \mathfrak{m}) is a local domain of mixed characteristic ($0 = \text{char}R, p = \text{char}R/\mathfrak{m}$). Prove that $(R^+)^{\wedge_p}$, the p -adic completion of R^+ , is perfectoid.

6. Now that R is a local domain of mixed characteristic ($0 = \text{char}R, p = \text{char}R/\mathfrak{m}$) and J is an ideal of R containing p . Suppose that $g \in R^+$. Prove that $g \in JR^+$ if and only if $g \in J(R^+)^{\wedge_p}$. This was sketched briefly in the lecture.

7. Now that $(R, \mathfrak{m} = (p = x_1, x_2, \dots, x_d))$ is a local domain of mixed characteristic ($0 = \text{char}R, p = \text{char}R/\mathfrak{m}$). Suppose that $g \in R^+$. Prove that

$$g \in (p, x_2, \dots, x_d)R^+$$

if and only if

$$g^{1/p} \in (p^{1/p}, x_2^{1/p}, \dots, x_d^{1/p})R^+$$

by using the previous exercise and the fact that $(R^+)^{\wedge_p}$ is perfectoid.

8. Consider $f = 2^3 + x^2 \in A = \mathbb{Z}_2[[x]]$ and prove $\text{ppt}(f) \leq 1/2$ directly. ¹

Hint: It suffices to find some $s \in A^+$ such that $f^{1/2} = 2s + x$. Try squaring both sides to prove the s exists.

¹The inequality $\text{ppt}(f) \geq 1/2$ can be proven via a deformation argument on the normalization of the extended Rees algebra $A[4t, xt, t^{-1}]$, similar to the one from Thursday's lecture. It follows that $\text{ppt}(f) = 1/2$.

9. On Tuesday, you computed a log resolution $Y \rightarrow \text{Spec}k[x, y]$ of $(k[x, y], y^2 - x^3)$. Compute the sheaf ω_Y for that map by using the following fact.

If you blowup a closed point on a 2-dimensional nonsingular variety (or scheme) X to obtain $f : Y \rightarrow X$ with exceptional curve E , then $\omega_Y = \mathcal{O}(E) \otimes f^*\omega_X$.

10. Using the Grothendieck duality arguments from Tuesday's exercises, plus trace arguments, one can prove the following in characteristic zero for $f \in R$ (again, R is regular) assuming $Y \rightarrow \text{Spec}R$ is a log resolution² of (R, f) .

$$\text{lct}(f) = \sup\{t > 0 \mid \Gamma(Y, \omega_Y \otimes \mathcal{O}_Y(\lfloor t \text{div}_Y(f) \rfloor)) = \omega_X\}.$$

Use this to compute the lct of $y^2 - x^3 \in k[x, y]$.

²A resolution of singularities such that the exceptional set is a divisor E so that $E + \text{div}_Y(f)$ is SNC.

II Macaulay2 exercises

Remember, you can run Macaulay2 online using the two links below if you don't have it installed on your computer.

- Macaulay2 at Georgia Tech
- Macaulay2 at University of Melbourne

II.1 F -pure thresholds in characteristic $p > 0$

Run the following commands.

```
needsPackage "TestIdeals"  
needsPackage "FrobeniusThresholds"
```

You can make a ring by the following command.

1. Make a ring and pick an element in it by running the following commands.

```
p = 5  
R = ZZ/p[x,y]  
f = x*y*(x-y)  
fpt(f)
```

Vary the p , make a conjecture about the value of $\text{fpt}(f)$ when $p \equiv 2 \pmod{3}$.

2. Do the same for $y^2 - x^3$.

II.2 Log canonical thresholds in characteristic zero

```
needsPackage "BernsteinSato"  
R = QQ[x,y,z]  
f = x^3+y^4+z^5
```

You can then run the following to compute the log canonical threshold (lct).

```
lct(ideal(f))
```

Again, these computations are *not* done by computing resolutions of singularities. These are done via D -module techniques.

We now have some exercises:

3. Do some experiments to make a prediction of what the lct of

$$x_1^{a_1} + \dots + x_n^{a_n}$$

is. (At least if you don't already know the answer).

4. Compare the speed of computing lct in characteristic 0 with computing fpt in characteristic $p > 0$.