Exercises, Lecture #2

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I Characteristic 0 singularities

1. Recall we blew up the origin in Spec$k[x,y,z]/(x^4+y^4+z^4)$ to obtain Y. Verify that $H^1(Y,\mathcal{O}_Y) \neq 0$ by a direct computation and hence conclude that $k[x,y,z]/(x^4+y^4+z^4)$ does not have rational singularities.

2. Compute a log resolution[1] of $(k[x,y], y^2-x^3)$. Let me do the first step.

3. Use the fact that rational singularities have the derived splinter property to prove the following theorem of Boutout. If $\phi : R \hookrightarrow S$ is a map of integral domains of finite type over a field of characteristic 0 such that ϕ is split as a map R-modules. If S has rational singularities, prove that so does R.

[1] Meaning the exceptional and the inverse of $V(f)$ are in simple normal crossings

4. We will prove *Kempf's criterion for rational singularities.* Namely, R, a domain of (essentially) finite type over a field of characteristic zero, has rational singularities if and only if the following two conditions hold:

   - R is Cohen-Macaulay.
   - For any (equivalently every) resolution of singularities $Y \to \mathrm{Spec} R$ we have $\Gamma(Y, \omega_Y) = \omega_R$ (here ω_R is a canonical module [2] and $\omega_Y = \bigwedge^{\dim Y} \Omega^1_{Y/k}$).

   Prove Kempf's criterion using the following two theorems.

   (a) Grothendieck duality over a Cohen-Macaulay ring.

   $$\mathbf{R}\Gamma(Y, \mathbf{R}\mathcal{H}om_Y(F, \omega_Y)) \cong \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma(Y, F), \omega_R)$$

   where F is a coherent sheaf on Y.

   (b) Grauert-Riemenschneider vanishing. $H^i(Y, \omega_Y) = 0$ for all $i > 0$.

5. Prove that regular rings of finite type over a field of characteristic zero have rational singularities. You may use without proof that if $\pi : Y \to X$ is a proper birational map between Noetherian non-singular varieties, then $K_Y - \pi^* K_X$ has non-negative coefficients.[3]

[2]if $R = S/I$ where S is a polynomial ring, or localization thereof, then $\omega_R \cong \mathrm{Ext}^{\dim S - \dim R}(R, S)$

[3]For a challenge, prove this directly.

II Macaulay2 exercises

Remember, you can run Macaulay2 online using the two links below if you don't have it installed on your computer.

- Macaulay2 at Georgia Tech
- Macaulay2 at University of Melbourne

II.1 Blowups and cohomology

You can compute cohomology of a blowup in Macaulay2. Try running the following code.

```
needsPackage "BGG"
S = QQ[x,y,z]
f = x^2+y*z
m = ideal(x,y,z)
T = reesAlgebra m
A = ambient T
J = saturate(ideal(sub(f, T)), sub(m, T))
N = A^1/((ideal T) + sub(J, A))
C = directImageComplex N
```

The complex `C` above is the complex $\mathbf{R}\Gamma(Y, \mathcal{O}_Y)$ we saw in the lecture. Note `J` is the ideal defining the strict transform in the blowup. We had to go to the ambient because that's how the package `BGG` likes it. You can compute `C`'s cohomology in each degree by the following.

```
HH^0(C)
HH^1(C)
HH^2(C)
ann HH^0(C)
ann HH^1(C)
degree HH^1(C)
```

The command `ann` computes the annihilator. The command `degree` computes the length of a finite length module.

1. Verify that $x^2 + y * z$ defines rational singularities in Macaulay2.

2. Verify that $x^3 + y^3 + z^3$ and $x^4 + y^4 + z^4$ define non-rational singularities.

3. Find the limits of this package.

II.2 Packages for working with singularities in char 0

Run the following commands after you `restart`:

```
needsPackage "BernsteinSato"
R = QQ[x,y,z]
f = x^3+y^4+z^5
```

You can then run the following to decide if $R/(f)$ has rational singularities.

```
hasRationalSing({f})
```

The function `hasRationalSing` is passed a list of generators of an ideal and it decides if the quotient has rational singularities. It only works if the list makes up a regular sequence. It is worth noting that these computations are *not* done by computing resolutions of singularities. These are done via D-module techniques.

II.3 Exercises

4. Compare the speed of determining whether a hypersurface has rational singularities in characteristic 0 or is F-regular in characteristic $p > 0$. You might find the command `elapsedTime` useful. As in

   ```
   elapsedTime hasRationalSing({f})
   elapsedTime isFRegular(R/ideal(g))
   ```

   Note `random(3, R)` makes a random element of degree 3 in a polynomial ring.

5. How does the speed of computing F-regularity change as one increases the characteristic?