Exercises, Lecture #1

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I Characteristic $p > 0$ singularities

1. Suppose that $R \hookrightarrow S$ is a map of rings that is split as a map of R-modules. Prove that $(IS) \cap R = I$ for every ideal I of R.

2. Prove that $R = \mathbb{F}_p[x^2, x^3]$ is not F-split by showing that $x^3 \in (x^2)R^{1/p}$.

3. Prove that $\mathbb{F}_p[x, y, z]/(x^d + y^d + z^d)$ is not F-split for any $d \geq 4$ by using the first exercise and proving that $(z^{d-1})^p \in (x^p, y^p)$.

4. Suppose R is an F-finite[1] Noetherian domain. Use Kunz' theorem to prove that the set of $Q \in \mathrm{Spec}R$ such that R_Q is regular is an open subset of $\mathrm{Spec}R$.

5. Suppose $R \to S$ is a map of Noetherian domains that splits as a map of R-modules.
   (a) If S is F-split, show that R is also F-split.
   (b) If S is strongly F-regular[2], show that R is strongly F-regular.

6. Suppose R is a Noetherian F-finite domain. Prove that the set of $Q \in \mathrm{Spec}R$ such that R_Q is F-split is open.

   Hint: Show that R is F-split if and only if $\mathrm{Hom}_R(R^{1/p}, R) \xrightarrow{\text{eval@1}} R$ is surjective.

[1] $R^{1/p}$ is a finite R-modules

[2] $\forall c \in S$, $0 \neq c$, $\exists e > 0$ such that $S \xrightarrow{1 \mapsto c^{1/p^e}} S^{1/p^e}$ splits

II Macaulay2 exercises

First, if you don't have Macaulay2 installed, you can install it by following instructions here.

Main Macaulay2 Page

Or you can run it online using the two links below run Macaulay2 online if you don't have it installed.

- Macaulay2 at Georgia Tech
- Macaulay2 at University of Melbourne

Run the following commands:

```
needsPackage "TestIdeals"
needsPackage "FrobeniusThresholds"
```

You can make a ring by the following command.

```
R = ZZ/7[x,y,z]/ideal(x^4+y^4+z^4)
```

1. Use commands like `isFPure R` (F-pure is another name for F-split it our context) and `isFRegular R` to check if various rings from todays lecture are F-split or F-regular.

2. Use Macaulay2 to help make a conjecture about when $\mathbb{F}_p[x_1, \dots, x_n]/(x_1^d + \cdots + x_n^d)$ is F-regular for various n, at least when p is sufficiently large.

3. Make a new ring and pick an element in it by running the following commands.

```
p = 5
R = ZZ/p[x,y]
f = x*y*(x-y)
fpt(f)
```

Vary the p, make a conjecture about the value of $\mathrm{fpt}(f)$ when $p \equiv 2 (\mathrm{mod} 3)$.